

REFLEXIVE FUNCTORS IN ALGEBRAIC GEOMETRY

PEDRO SANCHO

ABSTRACT. Reflexive functors of modules naturally appear in Algebraic Geometry. In this paper we define a wide and elementary family of reflexive functors of modules, closed by tensor products and homomorphisms, in which Algebraic Geometry can be developed.

1. INTRODUCTION

Let X be a scheme over a field K . We can regard X as a covariant functor of sets from the category of commutative K -algebras to the category of sets through its functor of points X^\cdot , defined by $X^\cdot(S) := \text{Hom}_{K\text{-sch}}(\text{Spec } S, X)$, for all commutative K -algebras S . If $X = \text{Spec } K[x_1, \dots, x_n]/(p_1, \dots, p_m)$ then

$$\begin{aligned} X^\cdot(S) &= \text{Hom}_{K\text{-sch}}(\text{Spec } S, X) = \text{Hom}_{K\text{-alg}}(K[x_1, \dots, x_n]/(p_1, \dots, p_m), S) \\ &= \{s \in S^n : p_1(s) = \dots = p_m(s) = 0\}. \end{aligned}$$

It is well known that $\text{Hom}_{K\text{-sch}}(X, Y) = \text{Hom}_{\text{funct.}}(X^\cdot, Y^\cdot)$, and X is a group K -scheme if and only if X^\cdot is a functor of groups.

We can regard K as functor of rings \mathcal{K} , by defining $\mathcal{K}(S) := S$, for all commutative K -algebras S . Let V be a K -vector space. We can regard V as a covariant functor of \mathcal{K} -modules, \mathcal{V} , by defining $\mathcal{V}(S) := V \otimes_K S$. We will say that \mathcal{V} is the \mathcal{K} -quasi-coherent module associated with V . If $V = \oplus_I K$ then $\mathcal{V}(S) = \oplus_I S$. The category of K -vector spaces, $\mathcal{C}_{K\text{-vect}}$, is equivalent to the category of quasi-coherent \mathcal{K} -modules, $\mathcal{C}_{\text{qs-coh } \mathcal{K}\text{-mod}}$: the functors $\mathcal{C}_{K\text{-vect}} \rightsquigarrow \mathcal{C}_{\text{qs-coh } \mathcal{K}\text{-mod}}$, $V \rightsquigarrow \mathcal{V}$ and $\mathcal{C}_{\text{qs-coh } \mathcal{K}\text{-mod}} \rightsquigarrow \mathcal{C}_{K\text{-vect}}$, $\mathcal{V} \rightsquigarrow \mathcal{V}(K)$ give the equivalence.

It is well known that the theory of linear representations of a group scheme G can be developed, via their associated functors, as a theory of an abstract group and its linear representations. Thus, if $G = \text{Spec } A$ is affine, the category of comodules over A is equivalent to the category of quasi-coherent G^\cdot -modules.

Given a functor of \mathcal{K} -modules, \mathbb{M} (that is, a covariant functor from the category of commutative K -algebras to the category of abelian groups, with a structure of \mathcal{K} -module), we denote $\mathbb{M}^* := \text{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{K})$ (see 2.2). We say that \mathbb{M} is a reflexive functor of modules if $\mathbb{M} = \mathbb{M}^{**}$.

If V is an infinite dimensional vector space, $V \not\cong V^{**}$. Moreover, if $G = \text{Spec } A$ is a group scheme and V is a comodule over A , V^* is not a comodule over A (as one would naively think). However, $\mathcal{V} = \mathcal{V}^{**}$ (see [2] and 2.7) and \mathcal{V}^* is a G^\cdot -module in the obvious way (see [4]). The category of comodules over A is not equivalent to the category of A^* -modules, but the category of comodules over A is equivalent to the category of quasi-coherent \mathcal{A}^* -modules.

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Corresponding Author: Pedro Sancho.

Let \mathbb{X} be a functor of sets and $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om_{\text{funct.}}(\mathbb{X}, \mathcal{K})$ the functor of functions of \mathbb{X} . We say that \mathbb{X} is an affine functor if $\mathbb{X} = \text{Spec } \mathbb{A}_{\mathbb{X}} := \mathbb{H}om_{\mathcal{K}\text{-alg}}(\mathbb{A}_{\mathbb{X}}, \mathcal{K})$ and $\mathbb{A}_{\mathbb{X}}$ is reflexive, see 7.1 and 7.2 for details (we warn the reader that in the literature affine functors are sometimes defined to be functors of points of affine schemes). The functors of points of affine schemes and formal schemes are affine functors (see 7.14). Let \mathbb{G} be an affine functor of monoids. $\mathbb{A}_{\mathbb{G}}^*$ is a functor of algebras and the category of \mathbb{G} -modules is equivalent to the category of $\mathbb{A}_{\mathbb{G}}^*$ -modules. Applications of these results include Cartier duality (see 8.17), neutral Tannakian duality for affine group schemes and the equivalence between formal groups and Lie algebras in characteristic zero (see [9]).

In summary, functors from the category of commutative algebras to the category of sets (groups, rings, etc.) naturally appear in Algebraic Geometry and many definitions and results are better understood in this language. Many results are based in the reflexivity of the considered functors.

Every reflexive functor is isomorphic to a functor of \mathcal{K} -submodules of $\prod_I \mathcal{K}$ (for some set I) and it is isomorphic to an inverse limit of \mathcal{K} -quasi-coherent modules, that is, it is pro-quasicoherent (see [9]). Is this family closed under tensor products? Is this family closed under homomorphisms? We do not know to answer these questions without adding hypothesis. In this paper we define a wide and elementary family of reflexive functors of modules, closed by tensor products and homomorphisms, in which Algebraic Geometry can be developed.

Now assume $K = R$ is a commutative ring. Let I be a set and $\alpha \subseteq I$, we denote $\mathcal{R}^{(\alpha)} = \bigoplus_{\alpha} \mathcal{R}$ and we have the obvious inclusion $\mathcal{R}^{(\alpha)} \times \mathcal{R}^{I-\alpha} \subseteq \mathcal{R}^{\alpha} \times \mathcal{R}^{I-\alpha} = \mathcal{R}^I$. Let $H_{\alpha} := \mathcal{R}^{(\alpha)} \oplus \mathcal{R}^{I-\alpha}$. We say that a functor of \mathcal{R} -modules, \mathbb{M} , is essentially free if there exist a subset I and a subset of the set of parts of I , P such that

$$\mathbb{M} \simeq \bigcap_{\alpha \in P} H_{\alpha}$$

Let $P^{\circ} := \{\beta \subseteq I : |\beta \cap \alpha| < \infty\}$. It is easy to check that $\bigcap_{\alpha \in P} H_{\alpha} = \bigcup_{\beta \in P^{\circ}} \mathcal{R}^{\beta}$ (where $\mathcal{R}^{\beta} = \mathcal{R}^{\beta} \times \{0\} \subseteq \mathcal{R}^{\beta} \times \mathcal{R}^{I-\beta} = \mathcal{R}^I$).

Let \mathfrak{F} be the category of essentially free \mathcal{R} -modules. We prove (see 3.8):

- (1) Essentially free functors of \mathcal{R} -modules are reflexive.
- (2) If M is a free R -module, then \mathcal{M} is essentially free.
- (3) Essentially free modules are pro-quasicoherent modules (see 3.12).
- (4) If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \in \mathfrak{F}$ and $(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**} \in \mathfrak{F}$, which satisfies

$$\mathbb{H}om_{\mathcal{R}}((\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**}, \mathbb{M}'') = \mathbb{H}om_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}', \mathbb{M}''),$$

for every reflexive functor of \mathcal{R} -modules, \mathbb{M}'' .

- (5) If $\mathbb{A}, \mathbb{A}' \in \mathfrak{F}$ are functors of \mathcal{R} -algebras, then $(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}')^{**} \in \mathfrak{F}$ is a functor of \mathcal{R} -algebras and

$$\mathbb{H}om_{\mathcal{R}\text{-alg}}((\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}')^{**}, \mathbb{A}'') = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}', \mathbb{A}''),$$

for every \mathcal{R} -algebra \mathbb{A}'' which is a reflexive functor of \mathcal{R} -modules.

- (6) If $\mathbb{A}, \mathbb{B} \in \mathfrak{F}$ are pro-quasicoherent functors of algebras, then $(\mathbb{A}^* \otimes_{\mathcal{R}} \mathbb{B}^*)^* \in \mathfrak{F}$ and it is a functor of pro-quasicoherent algebras, which satisfies

$$\mathbb{H}om_{\mathcal{R}\text{-alg}}((\mathbb{A}^* \otimes_{\mathcal{R}} \mathbb{B}^*)^*, \mathbb{C}) = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{B}, \mathbb{C}),$$

for every pro-quasicoherent functor of algebras, \mathbb{C} (see 6.7).

- (7) If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, the natural morphism $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \rightarrow \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R))$ is injective. If $\mathbb{A} \in \mathfrak{F}$ is a functor of \mathcal{R} -algebras and $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$ are functors of \mathbb{A} -modules, then a morphism of \mathcal{R} -modules $\mathbb{M} \rightarrow \mathbb{M}'$ is a morphism of \mathbb{A} -modules if and only if $\mathbb{M}(R) \rightarrow \mathbb{M}'(R)$ is a morphism of $\mathbb{A}(R)$ -modules. Let M be an R -module. If \mathcal{M} is an \mathbb{A} -module, then the set of all quasi-coherent \mathbb{A} -submodules of \mathcal{M} is equal to the set of all $\mathbb{A}(R)$ -submodules of M (see 4.1, 4.5 and 4.2). These results have obvious applications to the theory of linear representations of functors of monoids (see 4.6).
- (7') Assume that K is a field and that \mathfrak{F} is the family of reflexive functors of \mathcal{K} -modules. Then, all the results of (7) are likewise true.

We prove that a functor of \mathcal{R} -modules \mathbb{C} is a functor of coalgebras if and only if \mathbb{C}^* is a functor of \mathcal{R} -algebras. In particular, an R -module C is a coalgebra if and only if C^* is a functor of \mathcal{R} -algebras.

We will say that a reflexive functor \mathbb{B} of pro-quasicoherent algebras is a functor of bialgebras if \mathbb{B}^* is a functor of algebras and the dual morphisms of the multiplication morphism $\mathbb{B}^* \otimes_{\mathcal{R}} \mathbb{B}^* \rightarrow \mathbb{B}^*$ and the unit morphism $\mathcal{R} \rightarrow \mathbb{B}^*$ are morphisms of functors of algebras (see 8.1).

B is an R -bialgebra (in the standard sense) if and only if \mathcal{B} is a functor of bialgebras (see Proposition 8.3).

In the literature, there have been many attempts to obtain a well-behaved duality for non finite dimensional bialgebras (see [10] and references therein). One of them, for example, states that the functor that associates with each bialgebra A over a field K the so-called dual bialgebra $A^\circ := \varinjlim_{I \in J} (A/I)^*$, where J is the set of bilateral

ideals $I \subset A$ such that $\dim_K A/I < \infty$, is auto-adjoint (see [1] and 8.9). Another one associates with each bialgebra A over a pseudocompact ring R the bialgebra A^* endowed with a certain topology (see [5, Exposé VII_B 2.2.1]).

In this paper (see 8.5), we prove the following theorem.

Theorem 1.1. *Let $\mathcal{C}_{\mathfrak{F}\text{-bialg}}$ be the category of pro-quasicoherent functors $\mathbb{B} \in \mathfrak{F}$ of bialgebras. The functor $\mathcal{C}_{\mathfrak{F}\text{-bialg}} \rightsquigarrow \mathcal{C}_{\mathfrak{F}\text{-bialg}}, \mathbb{B} \rightsquigarrow \mathbb{B}^*$ is a categorical anti-equivalence.*

Finally, let us comment some geometric aspects of this theory. Let $\mathbb{A} \in \mathfrak{F}$ be a functor of algebras. We prove that $\mathrm{Spec} \mathbb{A}$ is a direct limit of affine schemes. Let \mathbb{X} be a functor of sets and assume $\mathbb{A}_{\mathbb{X}} \in \mathfrak{F}$. \mathbb{X} is affine if and only if $\mathbb{A}_{\mathbb{X}}$ is a functor of pro-quasicoherent algebras (see 7.8). Classical formal schemes correspond to functors $\mathrm{Spec} C^*$ (where C is a functor of algebras or equivalently C is a coalgebra). We prove that the category of affine functors of monoids \mathbb{G} is anti-equivalent to the category of functors of commutative bialgebras. In particular, Cartier duality is obtained (see 8.17).

2. PRELIMINARIES

Let R be a commutative ring (associative with a unit). All functors considered in this paper are covariant functors from the category of commutative R -algebras (always assumed to be associative with a unit) to the category of sets. A functor \mathbb{X} is said to be a functor of sets (resp. groups, rings, etc.) if \mathbb{X} is a functor from the category of commutative R -algebras to the category of sets (resp. groups, rings, etc.).

Notation 2.1. *For simplicity, given a functor of sets \mathbb{X} , we sometimes use $x \in \mathbb{X}$ to denote $x \in \mathbb{X}(S)$. Given $x \in \mathbb{X}(S)$ and a morphism of commutative R -algebras $S \rightarrow S'$, we still denote by x its image by the morphism $\mathbb{X}(S) \rightarrow \mathbb{X}(S')$.*

Let \mathcal{R} be the functor of rings defined by $\mathcal{R}(S) := S$, for all commutative R -algebras S . A functor of sets \mathbb{M} is said to be a functor of \mathcal{R} -modules if we have morphisms of functors of sets, $\mathbb{M} \times \mathbb{M} \rightarrow \mathbb{M}$ and $\mathcal{R} \times \mathbb{M} \rightarrow \mathbb{M}$, so that $\mathbb{M}(S)$ is an S -module, for every commutative R -algebra S .

Let \mathbb{M} and \mathbb{M}' be functors of \mathcal{R} -modules. A morphism of functors of \mathcal{R} -modules $f: \mathbb{M} \rightarrow \mathbb{M}'$ is a morphism of functors such that the defined morphisms $f_S: \mathbb{M}(S) \rightarrow \mathbb{M}'(S)$ are morphisms of S -modules, for all commutative R -algebras S . We will denote by $\text{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ the family of all morphisms of \mathcal{R} -modules from \mathbb{M} to \mathbb{M}' .

Given a commutative R -algebra S , we denote by $\mathbb{M}|_S$ the functor \mathbb{M} restricted to the category of commutative S -algebras. We will denote by $\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')^1$ the functor of \mathcal{R} -modules

$$\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')(S) := \text{Hom}_S(\mathbb{M}|_S, \mathbb{M}'|_S).$$

Obviously,

$$(\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}'))|_S = \mathbb{H}\text{om}_S(\mathbb{M}|_S, \mathbb{M}'|_S).$$

Notation 2.2. *We denote $\mathbb{M}^* = \mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$.*

Notation 2.3. *Tensor products, direct limits, inverse limits, etc., of functors of \mathcal{R} -modules and kernels, cokernels, images, etc., of morphisms of functors of \mathcal{R} -modules are regarded in the category of functors of \mathcal{R} -modules.*

We have that

$$\begin{aligned} (\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')(S) &= \mathbb{M}(S) \otimes_S \mathbb{M}'(S), \quad (\text{Ker } f)(S) = \text{Ker } f_S, \quad (\text{Coker } f)(S) = \text{Coker } f_S, \\ (\text{Im } f)(S) &= \text{Im } f_S, \quad (\varinjlim_{i \in I} \mathbb{M}_i)(S) = \varinjlim_{i \in I} (\mathbb{M}_i(S)), \quad (\varprojlim_{i \in I} \mathbb{M}_i)(S) = \varprojlim_{i \in I} (\mathbb{M}_i(S)). \end{aligned}$$

Definition 2.4. *Given an R -module M (resp. N , etc.), \mathcal{M} (resp. \mathcal{N} , etc.) will denote the functor of \mathcal{R} -modules defined by $\mathcal{M}(S) := M \otimes_R S$ (resp. $\mathcal{N}(S) := N \otimes_R S$, etc.). \mathcal{M} will be called quasi-coherent \mathcal{R} -module (associated with M).*

Proposition 2.5. [2, 1.3] *For every functor of \mathcal{R} -modules \mathbb{M} and every R -module M , it is satisfied that*

$$\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}) = \text{Hom}_R(M, \mathbb{M}(R)).$$

The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of R -modules and the category of quasi-coherent \mathcal{R} -modules ([2, 1.12]). In particular, $\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \text{Hom}_R(M, M')$. For any pair of R -modules M and N , the quasi-coherent module associated with $M \otimes_R N$ is $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$. $\mathcal{M}|_S$ is the quasi-coherent S -module associated with $M \otimes_R S$.

Proposition 2.6. [2, 1.8] *Let M and M' be R -modules. Then,*

$$\mathbb{H}\text{om}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}'.$$

If $\mathcal{M}' = \mathcal{R}$, in the previous proposition, we obtain the following theorem.

¹In this paper, we will only consider well defined functors $\mathbb{H}\text{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$, that is to say, functors such that $\text{Hom}_S(\mathbb{M}|_S, \mathbb{M}'|_S)$ is a set, for all S .

Theorem 2.7. [2, 1.10] *Let M be an R -module. Then*

$$\mathcal{M}^{**} = \mathcal{M}.$$

A functor of \mathcal{R} -modules \mathcal{M}^* is a quasi-coherent \mathcal{R} -module if and only if M is a projective finitely generated R -module (see [3]).

Notation 2.8. *Let $i: R \rightarrow S$ be a ring homomorphism between commutative rings. Given a functor of \mathcal{R} -modules, \mathbb{M} , let $i^*\mathbb{M}$ be the functor of \mathcal{S} -modules defined by $(i^*\mathbb{M})(S') := \mathbb{M}(S')$. Given a functor of \mathcal{S} -modules, \mathbb{M}' , let $i_*\mathbb{M}'$ be the functor of \mathcal{R} -modules defined by $(i_*\mathbb{M}')(R') := \mathbb{M}(S \otimes_R R')$.*

Adjunction formula 2.9. [2, 1.12] *Let \mathbb{M} be a functor of \mathcal{R} -modules and let \mathbb{M}' be a functor of \mathcal{S} -modules. Then,*

$$\mathrm{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{M}') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{M}').$$

Corollary 2.10. *Let \mathbb{M} be a functor of \mathcal{R} -modules. Then*

$$\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, S),$$

for all commutative R -algebras S .

Proof. $\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}|_S, S) \stackrel{2.9}{=} \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, S)$. □

Definition 2.11. *Let \mathbb{M} be a functor of \mathcal{R} -modules. We will say that \mathbb{M}^* is a dual functor. We will say that a functor of \mathcal{R} -modules \mathbb{M} is reflexive if $\mathbb{M} = \mathbb{M}^{**}$.*

Examples 2.12. *Quasi-coherent modules and module schemes are reflexive functors of \mathcal{R} -modules.*

Proposition 2.13. *Let \mathbb{M} be a functor of \mathcal{R} -modules such that \mathbb{M}^* is a reflexive functor. The closure of dual functors of \mathcal{R} -modules of \mathbb{M} is \mathbb{M}^{**} , that is, we have the functorial equality*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}^{**}, \mathbb{M}'),$$

for every dual functor of \mathcal{R} -modules \mathbb{M}' .

Proof. Write $\mathbb{M}' = \mathbb{N}^*$. Then, $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M} \otimes \mathbb{N}, \mathcal{R}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{N} \otimes \mathbb{M}^{**}, \mathcal{R}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}^{**}, \mathbb{M}')$. □

A functor of rings (associative with a unit), \mathbb{A} , is said to be a functor of \mathcal{R} -algebras if we have a morphism of functors of rings $\mathcal{R} \rightarrow \mathbb{A}$ (and $\mathcal{R}(S) = S$ commutes with all elements of $\mathbb{A}(S)$, for every commutative R -algebra S).

Proposition 2.14. *Let \mathbb{A} be a functor of \mathcal{R} -algebras such that \mathbb{A}^* is a reflexive functor of \mathcal{R} -modules. The closure of dual functors of \mathcal{R} -algebras of \mathbb{A} is \mathbb{A}^{**} , that is, we have the functorial equality*

$$\mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathbb{B}) = \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}^{**}, \mathbb{B}),$$

for every functor of \mathcal{R} -algebras \mathbb{B} , such that \mathbb{B} is a dual functor of \mathcal{R} -modules.

*As a consequence, the category of dual functors of \mathbb{A} -modules is equal to the category of dual functors of \mathbb{A}^{**} -modules.*

Proof. Given a dual functor of \mathcal{R} -modules \mathbb{M}^* ,

$$\begin{aligned} \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \cdot^n \otimes \mathbb{A}, \mathbb{M}^*) &= \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \cdot^{n-1} \otimes \mathbb{A}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}, \mathbb{M}^*)) \\ &\stackrel{2.13}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \cdot^{n-1} \otimes \mathbb{A}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**}, \mathbb{M}^*)) \\ &\stackrel{\text{Ind.Hyp.}}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes \cdot^{n-1} \otimes \mathbb{A}^{**}, \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**}, \mathbb{M}^*)) \\ &= \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes \cdot^n \otimes \mathbb{A}^{**}, \mathbb{M}^*), \end{aligned}$$

by induction on n . Let $i: \mathbb{A} \rightarrow \mathbb{A}^{**}$ be the natural morphism. The multiplication morphism $m: \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A}$ defines a unique morphism $m': \mathbb{A}^{**} \otimes \mathbb{A}^{**} \rightarrow \mathbb{A}^{**}$ such that the diagram

$$\begin{array}{ccc} \mathbb{A} \otimes \mathbb{A} & \xrightarrow{i \otimes i} & \mathbb{A}^{**} \otimes \mathbb{A}^{**} \\ \downarrow m & & \downarrow m' \\ \mathbb{A} & \xrightarrow{i} & \mathbb{A}^{**} \end{array}$$

is commutative, because $\mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes \mathbb{A}, \mathbb{A}^{**}) = \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes \mathbb{A}^{**}, \mathbb{A}^{**})$. It follows easily that the algebra structure of \mathbb{A} defines an algebra structure on \mathbb{A}^{**} . Let us only check that m' satisfies the associative property: The morphisms $m' \circ (m' \otimes \text{Id})$, $m' \circ (\text{Id} \otimes m')$: $\mathbb{A}^{**} \otimes \mathbb{A}^{**} \otimes \mathbb{A}^{**} \rightarrow \mathbb{A}^{**}$ are equal because

$$\begin{aligned} (m' \circ (m' \otimes \text{Id})) \circ (i \otimes i \otimes i) &= m' \circ (i \otimes i) \circ (m \otimes \text{Id}) = i \circ m \circ (m \otimes \text{Id}) \\ &= i \circ m \circ (\text{Id} \otimes m) = m' \circ (i \otimes i) \circ (\text{Id} \otimes m) = (m' \circ (\text{Id} \otimes m')) \circ (i \otimes i \otimes i). \end{aligned}$$

The kernel of the morphism

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{A}, \mathbb{B}) \rightarrow \mathbb{H}om_{\mathcal{R}}(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}, \mathbb{B}), \quad f \mapsto f \circ m - m \circ (f \otimes f),$$

coincides with the kernel of the morphism

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**}, \mathbb{B}) \rightarrow \mathbb{H}om_{\mathcal{R}}(\mathbb{A}^{**} \otimes_{\mathcal{R}} \mathbb{A}^{**}, \mathbb{B}), \quad f \mapsto f \circ m' - m \circ (f \otimes f).$$

Then, $\mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathbb{B}) = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}^{**}, \mathbb{B})$.

Finally, given a dual functor of \mathcal{R} -modules \mathbb{M}^* , then $\text{End}_{\mathcal{R}} \mathbb{M}^* = (\mathbb{M}^* \otimes \mathbb{M})^*$ is a dual functor of \mathcal{R} -modules and

$$\mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \text{End}_{\mathcal{R}} \mathbb{M}^*) = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}^{**}, \text{End}_{\mathcal{R}} \mathbb{M}^*).$$

Given two \mathbb{A}^{**} -modules, \mathbb{M}' and \mathbb{M}^* , and a morphism $f: \mathbb{M}' \rightarrow \mathbb{M}^*$ of \mathbb{A} -modules, then f is a morphism of \mathbb{A}^{**} -modules because given $m' \in \mathbb{M}'$, the morphism $\mathbb{A}^{**} \rightarrow \mathbb{M}^*$, $a \mapsto f(am')$ is zero because $\mathbb{A} \rightarrow \mathbb{M}^*$, $a \mapsto f(am') - af(m')$ is zero. Then, the category of dual functors of \mathbb{A} -modules is equal to the category of dual functors of \mathbb{A}^{**} -modules. \square

Notation 2.15. Given a functor of sets \mathbb{X} , the functor $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om(\mathbb{X}, \mathcal{R})$ is said to be the functor of functions of \mathbb{X} .

Example 2.16. Let $X = \text{Spec } A$ be an affine R -scheme and let X^\cdot be its functor of points, that is,

$$X^\cdot(S) := \text{Hom}_{R\text{-sch}}(\text{Spec } S, X) = \text{Hom}_{R\text{-alg}}(A, S).$$

Then, $\mathbb{A}_{X^\cdot} = \mathbb{H}om(X^\cdot, \mathcal{R}) = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathcal{R}[x], \mathcal{A}) = \mathcal{A}$.

Definition 2.17. Let \mathbb{X} be a functor of sets. Let $\mathcal{R}\mathbb{X}$ be the functor of \mathcal{R} -modules defined by

$$\mathcal{R}\mathbb{X}(S) := \oplus_{\mathbb{X}(S)} S = \{\text{formal finite } S\text{-linear combinations of elements of } \mathbb{X}(S)\}$$

Obviously, $\mathbb{H}\text{om}(\mathbb{X}, \mathbb{M}) = \mathbb{H}\text{om}_{\mathcal{R}}(\mathcal{R}\mathbb{X}, \mathbb{M})$, for all functors of \mathcal{R} -modules, \mathbb{M} .

Observe that $\mathbb{A}_{\mathbb{X}} = \mathbb{H}\text{om}(\mathbb{X}, \mathcal{R}) = (\mathcal{R}\mathbb{X})^*$.

Proposition 2.18. Let \mathbb{X} be a functor of sets and assume $\mathbb{A}_{\mathbb{X}} = \mathbb{B}^*$. Then,

$$\text{Hom}(\mathbb{X}, \mathbb{M}^*) = \text{Hom}_{\mathcal{R}}(\mathbb{B}, \mathbb{M}^*),$$

for every dual functor of \mathcal{R} -modules \mathbb{M}^* . In particular, if $\mathbb{A}_{\mathbb{X}}$ is reflexive, then

$$\text{Hom}(\mathbb{X}, \mathbb{M}^*) = \text{Hom}_{\mathcal{R}}(\mathbb{A}_{\mathbb{X}}^*, \mathbb{M}^*),$$

Proof. We have that

$$\begin{aligned} \text{Hom}(\mathbb{X}, \mathbb{M}^*) &= \text{Hom}_{\mathcal{R}}(\mathcal{R}\mathbb{X}, \mathbb{M}^*) = \text{Hom}_{\mathcal{R}}(\mathcal{R}\mathbb{X} \otimes_{\mathcal{R}} \mathbb{M}, \mathcal{R}) = \text{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{A}_{\mathbb{X}}) \\ &= \text{Hom}_{\mathcal{R}}(\mathbb{B} \otimes_{\mathcal{R}} \mathbb{M}, \mathcal{R}) = \text{Hom}_{\mathcal{R}}(\mathbb{B}, \mathbb{M}^*). \end{aligned}$$

□

Let \mathbb{G} be a functor of monoids. $\mathcal{R}\mathbb{G}$ is obviously a functor of \mathcal{R} -algebras. Given a functor of \mathcal{R} -algebras \mathbb{B} , it is easy to check the equality

$$\text{Hom}_{\text{mon}}(\mathbb{G}, \mathbb{B}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{R}\mathbb{G}, \mathbb{B}).$$

Theorem 2.19. Let \mathbb{G} be a functor of monoids with a reflexive functor of functions. Then, the closure of dual functors of algebras of \mathbb{G} is $\mathbb{A}_{\mathbb{G}}^*$. That is,

$$\text{Hom}_{\text{mon}}(\mathbb{G}, \mathbb{B}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{R}\mathbb{G}, \mathbb{B}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}_{\mathbb{G}}^*, \mathbb{B}),$$

for every dual functor of \mathcal{R} -algebras \mathbb{B} .

The category of quasi-coherent \mathbb{G} -modules is equivalent to the category of quasi-coherent $\mathbb{A}_{\mathbb{G}}^*$ -modules. Likewise, the category of dual functors of \mathbb{G} -modules is equivalent to the category of dual functors of $\mathbb{A}_{\mathbb{G}}^*$ -modules.

Proof. $(\mathcal{R}\mathbb{G})^* = \mathbb{A}_{\mathbb{G}}$ is reflexive. By Proposition 2.14, it is easy to complete the proof. □

Remark that the structure of functor of algebras of $\mathbb{A}_{\mathbb{G}}^*$ is the only one that makes the morphism $\mathbb{G} \rightarrow \mathbb{A}_{\mathbb{G}}^*$ a morphism of functors of monoids.

Example 2.20. Let $G = \text{Spec } A$ be an R -group scheme. The category of G -modules (that is to say, the category of comodules over A) is equivalent to the category of \mathcal{R} -quasi-coherent $\mathcal{R}[G]$ -modules. Then, the category of G -modules is equivalent to the category of \mathcal{R} -quasi-coherent \mathcal{A}^* -modules.

3. ESSENTIALLY FREE \mathcal{R} -MODULES

Notation 3.1. Let P be a subset of the set of parts of a set I . Denote $P^\circ := \{\beta \subseteq I : \beta \cap \alpha \text{ is a finite set for all } \alpha \in P\}$.

If $P \subseteq P_2$ then $P_2^\circ \subseteq P^\circ$. Obviously, $P \subseteq P^{\circ\circ}$. Then, $P^\circ \subseteq (P^\circ)^{\circ\circ} = (P^{\circ\circ})^\circ \subseteq P^\circ$. Hence,

$$P^\circ = P^{\circ\circ\circ}.$$

Given a set α , we denote $\mathcal{R}^\alpha = \prod_{\alpha} \mathcal{R}$ and $\mathcal{R}^{(\alpha)} = \oplus_{\alpha} \mathcal{R}$. If $\alpha_1 \subseteq \alpha_2$, we have the obvious epimorphism $\mathcal{R}^{(\alpha_2)} \rightarrow \mathcal{R}^{(\alpha_1)}$, $(\lambda_i)_{i \in \alpha_2} \mapsto (\lambda_i)_{i \in \alpha_1}$ and the obvious injective

morphism $\mathcal{R}^{\alpha_1} \rightarrow \mathcal{R}^{\alpha_2}$, $(\lambda_i)_{i \in \alpha_1} \mapsto (\lambda_i)_{i \in \alpha_1} := (\mu_i)_{i \in \alpha_2}$, where $\mu_i = \lambda_i$ if $i \in \alpha_1$ and $\mu_i = 0$ if $i \notin \alpha_1$.

Theorem 3.2. *Let P be a subset of the set of parts of I , such that if $\alpha, \alpha' \in P$ then $\alpha \cup \alpha' \in P$ and assume, for simplicity, $\cup_{\alpha \in P} \alpha = I$. Then,*

$$(1) \lim_{\alpha \in P} \mathcal{R}^\alpha = \lim_{\beta \in P^\circ} \mathcal{R}^\beta \subseteq \prod_I \mathcal{R}. \quad \lim_{\alpha \in P} \mathcal{R}^\alpha = \bigcap_{\alpha \in P} \mathcal{R}^\alpha \times \mathcal{R}^{I-\alpha}.$$

$$(2) (\lim_{\alpha \in P} \mathcal{R}^\alpha)^* = \lim_{\beta \in P^\circ} \mathcal{R}^\beta \text{ and } \lim_{\alpha \in P} \mathcal{R}^\alpha \text{ is reflexive. Then,}$$

$$\lim_{\alpha \in P} \mathcal{R}^\alpha = \lim_{\alpha \in P^{\circ\circ}} \mathcal{R}^\alpha.$$

Proof. 1. It is easy to prove that $\lim_{\alpha \in P} \mathcal{R}^\alpha = \{(\lambda_i) \in \mathcal{R}^I : \text{for each } \alpha \in P, \lambda_i = 0 \text{ for all } i \in \alpha \text{ but a finite number of } i\}$. Let $m = (\lambda_i) \in \mathcal{R}^I$ and $\beta = \{i \in I : \lambda_i \neq 0\}$. Then, $m \in \lim_{\alpha \in P} \mathcal{R}^\alpha$ if and only if $|\beta \cap \alpha| < \infty$, for all $\alpha \in P$. Hence,

$$\lim_{\alpha \in P} \mathcal{R}^\alpha = \lim_{\beta \in P^\circ} \mathcal{R}^\beta.$$

$$\lim_{\alpha \in P} \mathcal{R}^\alpha = \{(\lambda_i) \in \mathcal{R}^I : \text{for each } \alpha \in P, \lambda_i = 0 \forall i \in \alpha \text{ but a finite number of } i\} =$$

$$\bigcap_{\alpha \in P} \{(\lambda_i) \in \mathcal{R}^I : \lambda_i = 0 \text{ for all } i \in \alpha \text{ but a finite number of } i\} = \bigcap_{\alpha \in P} \mathcal{R}^\alpha \times \mathcal{R}^{I-\alpha}.$$

$$2. (\lim_{\alpha \in P} \mathcal{R}^\alpha)^* = (\lim_{\beta \in P^\circ} \mathcal{R}^\beta)^* = \lim_{\beta \in P^\circ} (\mathcal{R}^\beta)^* = \lim_{\beta \in P^\circ} \mathcal{R}^\beta. \text{ Now,}$$

$$(\lim_{\alpha \in P} \mathcal{R}^\alpha)^{**} = \lim_{\alpha \in P^{\circ\circ}} \mathcal{R}^\alpha = \lim_{\alpha \in P^{\circ\circ\circ}} \mathcal{R}^\alpha = \lim_{\alpha \in P^\circ} \mathcal{R}^\alpha = \lim_{\alpha \in P} \mathcal{R}^\alpha.$$

□

Observation 3.3. *Let P be the set of all numerable subsets of I . Then P° is the set of all finite subsets of I and $P^{\circ\circ}$ is the set of all subsets of I . Hence, $(\lim_{\alpha \in P} \mathcal{R}^\alpha)^{**} = (\lim_{\alpha \in P} \mathcal{R}^\alpha)^* = \lim_{\alpha \in P^\circ} \mathcal{R}^\alpha = \lim_{\alpha \in P^{\circ\circ}} \mathcal{R}^\alpha = \prod_I \mathcal{R}$. Then, if I is not numerable, $\lim_{\alpha \in P} \mathcal{R}^\alpha \subsetneq (\lim_{\alpha \in P} \mathcal{R}^\alpha)^{**}$. Moreover, $(\lim_{\alpha \in P} \mathcal{R}^\alpha)^* \neq \lim_{\alpha \in P} (\mathcal{R}^\alpha)^*$, because the latter term is not a reflexive functor of modules.*

Theorem 3.4. *Let L be a set of indices. For each $l \in L$, let P_l be a subset of the set of parts of a set I_l , such that if $\alpha, \alpha' \in P_l$ then $\alpha \cup \alpha' \in P_l$. Let $Q := \{(\alpha_l)_{l \in L} \in \prod_{l \in L} P_l : \alpha_l = \emptyset \text{ for all } l \in L \text{ except for a finite number of } l \in L\}$. Consider $\alpha = (\alpha_l)_{l \in L}$ as a subset of $\prod_{l \in L} I_l$ as follows $\alpha = \prod_{l \in L} \alpha_l$. Denote $\mathbb{M}_l := \lim_{\alpha_l \in P_l} \mathcal{R}^{\alpha_l}$ for each $l \in L$. Then,*

$$\prod_{l \in L} \mathbb{M}_l = \lim_{\alpha \in Q} \mathcal{R}^\alpha \text{ and } (\prod_{l \in L} \mathbb{M}_l)^* = \oplus_{l \in L} \mathbb{M}_l^*.$$

$$\text{Proof. } \prod_{l \in L} \mathbb{M}_l = \lim_{J \subset L, |J| < \infty} \prod_{j \in J} \mathbb{M}_j = \lim_{J \subset L, |J| < \infty} \lim_{\alpha \in \prod_{j \in J} P_j} \mathcal{R}^\alpha = \lim_{\alpha \in Q} \mathcal{R}^\alpha.$$

For the last statement, we can assume $P_l = P_l^{\circ\circ}$, by 3.2 2. Observe that $Q^\circ = \prod_{l \in L} P_l^\circ$ and that $Q^{\circ\circ} = Q$. Then,

$$\left(\prod_{l \in L} \mathbb{M}_l\right)^* = \lim_{\substack{\leftarrow \\ \alpha \in Q^\circ}} \mathcal{R}^{(\alpha)} \underset{Q=Q^{\circ\circ}}{=} \lim_{\substack{\rightarrow \\ \alpha \in Q}} \mathcal{R}^\alpha = \lim_{\substack{\rightarrow \\ \alpha \in Q}} (\oplus_{l \in L} \mathcal{R}^{\alpha_l}) = \oplus_{l \in L} \mathbb{M}_l^*.$$

□

Theorem 3.5. *Let P be a subset of the set of parts of I , such that if $\alpha, \alpha' \in P$ then $\alpha \cup \alpha' \in P$ and assume, for simplicity, $\cup_{\alpha \in P} \alpha = I$. Let Q be a subset of the set of parts of J , such that if $\beta, \beta' \in Q$ then $\beta \cup \beta' \in Q$ and assume, for simplicity, $\cup_{\beta \in Q} \beta = J$. Then,*

$$\begin{aligned} (1) \quad & \mathbb{H}om_{\mathcal{R}}\left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)}, \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right) = \lim_{\substack{\leftarrow \\ (\alpha, \beta) \in P^\circ \times Q}} \mathcal{R}^{(\alpha \times \beta)}. \\ (2) \quad & \left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)} \otimes_{\mathcal{R}} \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right)^* = \lim_{\substack{\leftarrow \\ \gamma \in P^\circ \times Q^\circ}} \mathcal{R}^{(\gamma)}. \text{ Hence,} \\ (a) \quad & \left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)} \otimes_{\mathcal{R}} \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right)^{**} = \lim_{\substack{\leftarrow \\ \gamma \in (P^\circ \times Q^\circ)^\circ}} \mathcal{R}^{(\gamma)}. \\ (b) \quad & \left(\left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)}\right)^* \otimes_{\mathcal{R}} \left(\lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right)^*\right)^* = \lim_{\substack{\leftarrow \\ \gamma \in P^{\circ\circ} \times Q^{\circ\circ}}} \mathcal{R}^{(\gamma)}. \end{aligned}$$

Proof.

$$\begin{aligned} 1. \quad & \mathbb{H}om_{\mathcal{R}}\left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)}, \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right) = \mathbb{H}om_{\mathcal{R}}\left(\lim_{\substack{\rightarrow \\ \alpha \in P^\circ}} \mathcal{R}^\alpha, \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right) \\ & = \lim_{\substack{\leftarrow \\ (\alpha, \beta) \in P^\circ \times Q}} \mathbb{H}om_{\mathcal{R}}(\mathcal{R}^\alpha, \mathcal{R}^{(\beta)}) = \lim_{\substack{\leftarrow \\ (\alpha, \beta) \in P^\circ \times Q}} \mathcal{R}^{(\alpha)} \otimes_{\mathcal{R}} \mathcal{R}^{(\beta)} \\ & = \lim_{\substack{\leftarrow \\ (\alpha, \beta) \in P^\circ \times Q}} \mathcal{R}^{(\alpha \times \beta)}. \\ 2. \quad & \left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)} \otimes_{\mathcal{R}} \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}\right)^* = \mathbb{H}om_{\mathcal{R}}\left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)} \otimes_{\mathcal{R}} \lim_{\substack{\leftarrow \\ \beta \in Q}} \mathcal{R}^{(\beta)}, \mathcal{R}\right) \\ & = \mathbb{H}om_{\mathcal{R}}\left(\lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)}, \lim_{\substack{\leftarrow \\ \beta \in Q^\circ}} \mathcal{R}^{(\beta)}\right) = \lim_{\substack{\leftarrow \\ \gamma \in P^\circ \times Q^\circ}} \mathcal{R}^{(\gamma)}. \end{aligned}$$

□

Definition 3.6. *We will say that a functor of \mathcal{R} -modules, \mathbb{M} , is essentially free if there exist a set I , a subset P of the set of parts of I (such that if $\alpha, \alpha' \in P$ then $\alpha \cup \alpha' \in P$) and an isomorphism of functors of \mathcal{R} -modules*

$$\mathbb{M} \simeq \lim_{\substack{\leftarrow \\ \alpha \in P}} \mathcal{R}^{(\alpha)}.$$

Let \mathfrak{F} be the category of essentially free \mathcal{R} -modules.

Examples 3.7. *If V is a free \mathcal{R} -module, \mathcal{V} and $\mathcal{V}^* \in \mathfrak{F}$.*

Theorem 3.8. (1) *Essentially free modules are reflexive.*

(2) *Let $\mathbb{M}_l \in \mathfrak{F}$, for all $l \in L$. Then, $\prod_{l \in L} \mathbb{M}_l$ and $\oplus_{l \in L} \mathbb{M}_l \in \mathfrak{F}$.*

(3) *If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, then $\mathbb{M}^*, \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \in \mathfrak{F}$.*

- (4) If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, then $(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^* \in \mathfrak{F}$ and $(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**}$ satisfies

$$\mathrm{Hom}_{\mathcal{R}}((\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**}, \mathbb{M}'') \stackrel{2.13}{=} \mathrm{Hom}_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}', \mathbb{M}''),$$

for every dual functor of \mathcal{R} -modules, \mathbb{M}'' .

- (5) If $\mathbb{A}, \mathbb{A}' \in \mathfrak{F}$ are functor of \mathcal{R} -algebras, then $(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}')^{**} \in \mathfrak{F}$ is a functor of \mathcal{R} -algebras and

$$\mathrm{Hom}_{\mathcal{R}\text{-alg}}((\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}')^{**}, \mathbb{A}'') \stackrel{2.14}{=} \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A}', \mathbb{A}''),$$

for every \mathcal{R} -algebra \mathbb{A}'' such that it is a dual functor of \mathcal{R} -modules.

Proof. It is consequence of 3.2, 3.4 and 3.5. \square

Lemma 3.9. *Let $\phi: \mathcal{R}^\alpha \rightarrow \mathcal{M}$ be a morphism of \mathcal{R} -modules. Then, $\mathrm{Im} \phi_R \subseteq M$ is a R -module of finite type and ϕ factors uniquely through an epimorphism onto the quasi-coherent module associated with $\mathrm{Im} \phi_R$.*

Proof. $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^\alpha, \mathcal{M}) = R^{(\alpha)} \otimes_R M$. Let $\{1_i\}_{i \in \alpha}$ be the standard basis of $R^{(\alpha)}$. Then, $\phi = 1_{i_1} \otimes m_1 + \cdots + 1_{i_n} \otimes m_n$, $N := \mathrm{Im} \phi_R = \langle m_1, \dots, m_n \rangle$ and ϕ factors uniquely through $1_{i_1} \otimes m_1 + \cdots + 1_{i_n} \otimes m_n \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^\alpha, \mathcal{N}) = R^{(\alpha)} \otimes_R N$. \square

Proposition 3.10. *$\mathcal{M} \in \mathfrak{F}$ if and only if M is a free R -module.*

Proof. If $\mathcal{M} \in \mathfrak{F}$ then $\mathcal{M} = \varinjlim_{\alpha \in P} \mathcal{R}^\alpha$. Consider the obvious injective morphism $i: \mathcal{R}^\alpha \rightarrow \mathcal{M}$. By Lemma 3.9, $\mathrm{Im} i_R$ is a R -module of finite type, then α is a finite set. Hence, $\mathcal{M} = \varinjlim_{\alpha \in P} \mathcal{R}^\alpha = \mathcal{R}^{(\cup_{\alpha \in P} \alpha)}$ and M is a free R -module. \square

Lemma 3.11. *Let $\mathbb{M} \in \mathfrak{F}$ and let M be an R -module. Then, every morphism of \mathcal{R} -modules $\phi: \mathbb{M} \rightarrow \mathcal{M}$ factors uniquely through an epimorphism onto the quasi-coherent module associated with the R -submodule $\mathrm{Im} \phi_R \subseteq M$.*

Proof. It is a consequence of Lemma 3.9 and the equality $\mathbb{M} = \varinjlim_{\alpha \in P} \mathcal{R}^\alpha$. \square

Theorem 3.12. *Let $\{\mathcal{M}_i\}_{i \in I}$ be the set of all quasi-coherent quotients of $\mathbb{M} \in \mathfrak{F}$. Then, $\mathbb{M}^* = \varinjlim_{i \in I} \mathcal{M}_i^*$. Therefore,*

$$\mathbb{M} = \varprojlim_{i \in I} \mathcal{M}_i.$$

Proof. Let S be a commutative R -algebra. $\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, S)$, by Corollary 2.10. The morphism $\varinjlim_{i \in I} \mathcal{M}_i^*(S) \rightarrow \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, S) = \mathbb{M}^*(S)$ is obviously injective,

and it is surjective by Lemma 3.11. Hence, $\mathbb{M}^* = \varinjlim_{i \in I} \mathcal{M}_i^*$ and $\mathbb{M} = \mathbb{M}^{**} =$

$$\varprojlim_{i \in I} \mathcal{M}_i.$$

\square

Remark 3.13. *Let $R = K$ be a field. Now assume \mathfrak{F} is the family of reflexive functors of \mathcal{K} -modules. Lemma 3.11 is true by [2, 2.13]. Hence, Theorem 3.12 is likewise true. Hence, every reflexive module is a direct limit of modules \mathcal{M}_i^* and it is an inverse limit of quasicohherent modules*

4. HOMOMORPHISMS AND APPLICATIONS

Proposition 4.1. *Let $\mathbb{M} \in \mathfrak{F}$ and let \mathbb{M}' be a dual functor of \mathcal{R} -modules. Then, the morphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \rightarrow \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R)), \phi \mapsto \phi_R$$

is injective..

Proof. $\mathbb{M} = \lim_{\substack{\rightarrow \\ \alpha \in P}} \mathcal{R}^\alpha$. It is sufficient to prove this proposition when $\mathbb{M} = \mathcal{R}^\alpha$. Now,

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^\alpha, \mathbb{N}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{N}, \mathcal{R}^\alpha) \subseteq \mathrm{Hom}_{\mathcal{R}}(\mathbb{N}, \mathcal{R}^\alpha) = \prod_{\alpha} \mathbb{N}^*(R).$$

Hence, the composite morphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^\alpha, \mathbb{M}') \rightarrow \mathrm{Hom}_R(\mathcal{R}^\alpha, \mathbb{M}'(R)) \rightarrow \prod_{\alpha} \mathbb{M}'(R)$ is injective. Then, the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^\alpha, \mathbb{M}') \rightarrow \mathrm{Hom}_R(\mathcal{R}^\alpha, \mathbb{M}'(R))$ is injective. \square

Proposition 4.2. *Let $\mathbb{A} \in \mathfrak{F}$ be a functor of \mathcal{R} -algebras, let $\mathbb{M} \in \mathfrak{F}$ be a functor of \mathbb{A} -modules and let \mathbb{M}' be a functor of \mathbb{A} -modules and a dual functor of \mathcal{R} -modules. Let $f: \mathbb{M} \rightarrow \mathbb{M}'$ be a morphism of \mathcal{R} -modules. Then, f is a morphism of \mathbb{A} -modules if and only if f_R is a morphism of $\mathbb{A}(R)$ -modules.*

Proof. The morphism f is a morphism of \mathbb{A} -modules if and only if $F: \mathbb{A} \otimes \mathbb{M} \rightarrow \mathbb{M}'$, $F(a \otimes m) := f(am) - af(m)$ is the zero morphism. Likewise, f_R is a morphism of $\mathbb{A}(R)$ -modules if and only if $F_R: \mathbb{A}(R) \otimes \mathbb{M}(R) \rightarrow \mathbb{M}'(R)$, $F_R(a \otimes m) = f_R(am) - af_R(m)$ is the zero morphism. Now, the proposition is a consequence of the injective morphisms,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}}(\mathbb{A} \otimes \mathbb{M}, \mathbb{M}') &= \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}, \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')) \xrightarrow[4.1]{\hookrightarrow} \mathrm{Hom}_R(\mathbb{A}(R), \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')) \\ &\xrightarrow[4.1]{\hookrightarrow} \mathrm{Hom}_R(\mathbb{A}(R), \mathrm{Hom}_R(\mathbb{M}(R), \mathbb{M}'(R))) = \mathrm{Hom}_R(\mathbb{A}(R) \otimes \mathbb{M}(R), \mathbb{M}'(R)). \end{aligned}$$

\square

Proposition 4.3. *Let $\mathbb{A} \in \mathfrak{F}$ be a functor of \mathcal{R} -algebras and let $\mathcal{M}, \mathcal{M}'$ be functors of \mathbb{A} -modules. Then,*

$$\mathrm{Hom}_{\mathbb{A}}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_{\mathbb{A}(R)}(M, M').$$

Proof. Proceed as in the proof of Proposition 4.2. \square

Notation 4.4. *Let M be an R -module and let $M' \subseteq M$ be an R -submodule. By abuse of notation we will say that \mathcal{M}' is a quasi-coherent submodule of \mathcal{M} .*

Proposition 4.5. *Let $\mathbb{A} \in \mathfrak{F}$ be a functor of \mathcal{R} -algebras, let \mathcal{M} be an \mathbb{A} -module and let $M' \subset M$ be an R -submodule. Then, \mathcal{M}' is a quasi-coherent \mathbb{A} -submodule of \mathcal{M} if and only if M' is an $\mathbb{A}(R)$ -submodule of M .*

Proof. Obviously, if \mathcal{M}' is an \mathbb{A} -submodule of \mathcal{M} then M' is an $\mathbb{A}(R)$ -submodule of M . Conversely, let us assume M' is an $\mathbb{A}(R)$ -submodule of M and let us consider the natural morphism of multiplication $\mathbb{A} \otimes_{\mathcal{R}} \mathcal{M}' \rightarrow \mathcal{M}$. By Lemma 3.11, the morphisms $\mathbb{A} \rightarrow \mathcal{M}$, $a \mapsto a \cdot m'$, for each $m' \in M'$, factors uniquely via \mathcal{M}' . Let

$\oplus_I R \xrightarrow{q} \oplus_J R \xrightarrow{p} M' \rightarrow 0$ be an exact sequence. Let i be the morphism $\mathcal{M}' \rightarrow \mathcal{M}$. There exists a (unique) morphism f' such that the diagram

$$\begin{array}{ccccc} \mathbb{A} \otimes_{\mathcal{R}} (\oplus_I \mathcal{R}) & \xrightarrow{Id \otimes q} & \mathbb{A} \otimes_{\mathcal{R}} (\oplus_J \mathcal{R}) & \xrightarrow{Id \otimes p} & \mathbb{A} \otimes_{\mathcal{R}} \mathcal{M}' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow \\ & & \mathcal{M}' & \xrightarrow{i} & \mathcal{M} \end{array}$$

is commutative. As $i_R \circ f'_R \circ (Id \otimes q)_R = 0$, then $f'_R \circ (Id \otimes q)_R = 0$. By Proposition 4.1, $f' \circ (Id \otimes q) = 0$. Hence, $\mathbb{A} \otimes_{\mathcal{R}} \mathcal{M}' \rightarrow \mathcal{M}$ factors through \mathcal{M}' .

$F: \mathbb{A} \otimes_{\mathcal{R}} \mathbb{A} \rightarrow \mathcal{M}'$, $F(a \otimes a') := a(a'm') - (aa')m'$ (for any $m' \in \mathcal{M}'$) is the zero morphism: F lifts to a (unique) morphism $\bar{F}: (\mathbb{A} \otimes_{\mathcal{R}} \mathbb{A})^{**} \rightarrow \mathcal{M}'$. Observe that $i \circ \bar{F} = 0$ because $i \circ F = 0$, then $\bar{F}_R = 0$ because i_R is injective. Finally, $\bar{F} = 0$ because it is determined by \bar{F}_R ; and $F = 0$. Likewise, $1 \cdot m' = m'$, for all $m' \in \mathcal{M}'$.

In conclusion, \mathcal{M}' is a quasi-coherent \mathbb{A} -submodule of \mathcal{M} . \square

Theorem 4.6. *Let \mathbb{G} be a functor of monoids and let \mathbb{M}, \mathbb{M}' be functors of \mathbb{G} -modules. Assume that $\mathbb{A}_{\mathbb{G}}, \mathbb{M}, \mathbb{M}' \in \mathfrak{F}$. Let $f: \mathbb{M} \rightarrow \mathbb{M}'$ be a morphism of \mathcal{R} -modules. Then, f is a morphism of \mathbb{G} -modules if and only if f_R is a morphism of $\mathbb{A}_{\mathbb{G}}^*(R)$ -modules.*

Let \mathcal{M} be a \mathbb{G} -module, then the set of quasi-coherent \mathbb{G} -submodules of \mathcal{M} is equal to the set of $\mathbb{A}_{\mathbb{G}}^(R)$ -submodules of \mathcal{M} .*

Proof. It is a consequence of Theorem 2.19, Proposition 4.2 and Proposition 4.5. \square

Remark 4.7. *Let $R = K$ be a field. Now assume \mathfrak{F} is the family of reflexive functors of K -modules. Recall Remark 3.13. All the propositions in this section are likewise true. If M' is a K -vector subspace, then $M' = \oplus_J K$. This fact simplifies the proof of Proposition 4.5.*

5. FUNCTOR OF COALGEBRAS

Notation 5.1. *Let $\mathbb{M}_1, \dots, \mathbb{M}_n$ be \mathcal{R} -modules, we denote*

$$\mathbb{M}_1 \tilde{\otimes}_{\mathcal{R}} \dots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n := (\mathbb{M}_1^* \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \mathbb{M}_n^*)^*.$$

Morphisms $\mathbb{M}_i \rightarrow \mathbb{N}_i$ induce an obvious morphism $\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n \rightarrow \mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_n$. Reader can check

- (1) $\mathcal{M}_1 \tilde{\otimes}_{\mathcal{R}} \dots \tilde{\otimes}_{\mathcal{R}} \mathcal{M}_n = \mathcal{M}_1 \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \mathcal{M}_n$.
- (2) $\mathcal{M}_1^* \tilde{\otimes}_{\mathcal{R}} \dots \tilde{\otimes}_{\mathcal{R}} \mathcal{M}_n^* = (\mathcal{M}_1 \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \mathcal{M}_n)^*$.
- (3) If $\mathbb{M}_1 \tilde{\otimes}_{\mathcal{R}} \dots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n$ and $\mathbb{N}_1 \tilde{\otimes}_{\mathcal{R}} \dots \tilde{\otimes}_{\mathcal{R}} \mathbb{N}_m$ are reflexive, then the obvious morphism $(\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n) \tilde{\otimes} (\mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m) \rightarrow \mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n \tilde{\otimes} \mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m$ is an isomorphism:

$$\begin{aligned} (\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n) \tilde{\otimes} (\mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m) &= \text{Hom}_{\mathcal{R}}((\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n)^*, (\mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m)^{**}) \\ &= \text{Hom}_{\mathcal{R}}((\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n)^*, \mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m) \\ &= \text{Hom}_{\mathcal{R}}((\mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n)^* \otimes \mathbb{N}_1^* \otimes \dots \otimes \mathbb{N}_m^*, \mathcal{R}) \\ &= \text{Hom}_{\mathcal{R}}(\mathbb{N}_1^* \otimes \dots \otimes \mathbb{N}_m^*, \mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n) = \mathbb{M}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{M}_n \tilde{\otimes} \mathbb{N}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{N}_m \end{aligned}$$

Definition 5.2. \mathbb{C} is said to be a functor of coalgebras if there exist a morphism of \mathcal{R} -modules $m': \mathbb{C} \rightarrow \mathbb{C} \tilde{\otimes}_{\mathcal{R}} \mathbb{C}$ coassociative (that is, $(m' \tilde{\otimes} Id) \circ m' = (Id \tilde{\otimes} m') \circ m'$, where $m' \tilde{\otimes} Id: \mathbb{C} \tilde{\otimes} \mathbb{C} \rightarrow \mathbb{C} \tilde{\otimes} \mathbb{C} \tilde{\otimes} \mathbb{C}$ and $Id \tilde{\otimes} m'$ are the obvious morphisms) and a counit (that is, a morphism of \mathcal{R} -modules $u': \mathbb{C} \rightarrow \mathcal{R}$ such that $(u' \tilde{\otimes} Id) \circ m' = Id = (Id \tilde{\otimes} u') \circ m'$, where $u' \tilde{\otimes} Id: \mathbb{C} \tilde{\otimes}_{\mathcal{R}} \mathbb{C} \rightarrow \mathcal{R} \tilde{\otimes}_{\mathcal{R}} \mathbb{C} = \mathbb{C}$ and $Id \tilde{\otimes} u'$ are the obvious morphisms).

Let C be an R -module. C is an R -coalgebra if and only if \mathcal{C} is a functor of coalgebras. Although we have called functor of coalgebras to \mathbb{C} , we warn the reader that $\mathbb{C}(S)$ is not necessarily an S -coalgebra

Proposition 5.3. \mathbb{C} is a functor of coalgebras if and only if \mathbb{C}^* is a functor of \mathcal{R} -algebras. In particular, an R -module C is a coalgebra if and only if \mathcal{C}^* is a functor of \mathcal{R} -algebras.

Proof. Observe that

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{C}^* \otimes \cdots \otimes \mathbb{C}^*, \mathbb{C}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{C}, (\mathbb{C}^* \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathbb{C}^*)^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{C}, \mathbb{C} \tilde{\otimes}_{\mathcal{R}} \cdots \tilde{\otimes}_{\mathcal{R}} \mathbb{C}).$$

Now, it is easy to complete the proof. \square

Notation 5.4. Let $\mathbb{M}_1, \dots, \mathbb{M}_n$ be \mathcal{R} -modules, we denote

$$\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n := (\mathbb{M}_1 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathbb{M}_n)^{**}.$$

Observe that

$$(\mathbb{M}_1 \tilde{\otimes}_{\mathcal{R}} \cdots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n)^* = \mathbb{M}_1^* \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n^*.$$

If \mathbb{M}_i is reflexive for all i , and $\mathbb{M}_1^* \tilde{\otimes}_{\mathcal{R}} \cdots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n^*$ is reflexive then

$$(\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n)^* = \mathbb{M}_1^* \tilde{\otimes}_{\mathcal{R}} \cdots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n^*.$$

If $\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n$ and $\mathbb{N}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{N}_m$ are reflexive, the natural morphism

$$\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n \bar{\otimes}_{\mathcal{R}} \mathbb{N}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{N}_m \rightarrow (\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n) \bar{\otimes}_{\mathcal{R}} (\mathbb{N}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{N}_m)$$

is an isomorphism (use Proposition 2.13).

Proposition 5.5. Let $\mathbb{M}_1, \dots, \mathbb{M}_n$ be essentially free \mathcal{R} -modules, and let us consider the natural morphism $\mathbb{M}_1^* \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathbb{M}_n^* \rightarrow (\mathbb{M}_1 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathbb{M}_n)^*$. Then, the dual morphism

$$\mathbb{M}_1 \bar{\otimes}_{\mathcal{R}} \cdots \bar{\otimes}_{\mathcal{R}} \mathbb{M}_n \rightarrow \mathbb{M}_1 \tilde{\otimes}_{\mathcal{R}} \cdots \tilde{\otimes}_{\mathcal{R}} \mathbb{M}_n$$

is injective.

Proof. Write $\mathbb{M}_j = \varprojlim_{\alpha \in P_j} R^\alpha \subseteq \mathcal{R}^{I_j}$, where P_j is a subset of the set of parts of I_j .

$(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n)^{**}$ and $(\mathbb{M}_1^* \otimes \cdots \otimes \mathbb{M}_n^*)^*$ are functors of \mathcal{R} -submodules of $\mathcal{R}^{\prod_{j \in J} I_j}$, by Theorem 3.2 and Theorem 3.5. \square

6. FUNCTORS OF PRO-QUASICOHERENT ALGEBRAS

Definition 6.1. Let A be an R -algebra. The associated functor \mathcal{A} is obviously a functor of \mathcal{R} -algebras. We will say that \mathcal{A} is a quasi-coherent \mathcal{R} -algebra.

Definition 6.2. We will say that a functor of \mathcal{R} -algebras is a functor of pro-quasicoherent algebras if it is the inverse limit of its quasi-coherent algebra quotients.

Examples 6.3. Quasi-coherent algebras are pro-quasicoherent functors of algebras.

Let $R = K$ be a field, A be a commutative K -algebra and $I \subseteq A$ be an ideal. Then, $\mathbb{B} = \varprojlim_{n \in \mathbb{N}} \mathcal{A}/\mathcal{I}^n \in \mathfrak{F}$ and it is a pro-quasicoherent algebra: $\mathbb{B} \simeq \prod_n \mathcal{I}^n/\mathcal{I}^{n+1}$, then $\mathbb{B}^* = \oplus_n (\mathcal{I}^n/\mathcal{I}^{n+1})^* = \varinjlim_n (\mathcal{A}/\mathcal{I}^n)^*$. Therefore, \mathbb{B}^* is equal to the direct limit of the dual of the quasi-coherent algebra quotients of \mathbb{B} . Dually, \mathbb{B} is a pro-quasicoherent algebra.

Proposition 6.4. Let $\mathbb{A} \in \mathfrak{F}$ and \mathbb{B} be functors of \mathcal{R} -algebras and assume that there exists an injective morphism of \mathcal{R} -modules $\mathbb{B} \hookrightarrow \mathcal{N}$. Then, any morphism of \mathcal{R} -algebras $\phi: \mathbb{A} \rightarrow \mathbb{B}$ factors uniquely through an epimorphism of algebras onto the quasi-coherent algebra associated with $\text{Im } \phi_R$.

Proof. By Lemma 3.11, the morphism $\phi: \mathbb{A} \rightarrow \mathbb{B}$ factors uniquely through an epimorphism $\phi': \mathbb{A} \rightarrow \mathcal{B}'$, where $\mathcal{B}' := \text{Im } \phi_R$. Obviously \mathcal{B}' is an R -subalgebra of $\mathbb{B}(R)$. We have to check that ϕ' is a morphism of functors of algebras.

Observe that if a morphism $f: \mathbb{A} \otimes \mathbb{A} \rightarrow \mathcal{N}$ factors through an epimorphism onto a quasi-coherent submodule \mathcal{N}' of \mathcal{N} then factors uniquely through \mathcal{N}' , because f and any morphism to \mathcal{N}' factors uniquely through $(\mathbb{A} \otimes \mathbb{A})^{**} \in \mathfrak{F}$.

Consider the diagram

$$\begin{array}{ccccc} \mathbb{A} \otimes \mathbb{A} & \xrightarrow{\phi' \otimes \phi'} & \mathcal{B}' \otimes \mathcal{B}' & \xrightarrow{i \otimes i} & \mathbb{B} \otimes \mathbb{B} \\ \downarrow m_{\mathbb{A}} & & \downarrow m_{\mathcal{B}'} & & \downarrow m_{\mathbb{B}} \\ \mathbb{A} & \xrightarrow{\phi'} & \mathcal{B}' & \xrightarrow{i} & \mathbb{B} \hookrightarrow \mathcal{N}, \end{array}$$

where $m_{\mathbb{A}}, m_{\mathcal{B}'}$ and $m_{\mathbb{B}}$ are the multiplication morphisms and i is the morphism induced by the morphism $\mathcal{B}' \rightarrow \mathbb{B}(R)$. We know $m_{\mathbb{B}} \circ (i \otimes i) \circ (\phi' \otimes \phi') = i \circ \phi' \circ m_{\mathbb{A}}$. The morphism $m_{\mathbb{B}} \circ (i \otimes i) \circ (\phi' \otimes \phi')$ factors uniquely onto \mathcal{B}' , more concretely, through $m_{\mathcal{B}'} \circ (\phi' \otimes \phi')$. The morphism $i \circ \phi' \circ m_{\mathbb{A}}$ factors uniquely onto \mathcal{B}' , effectively, through $\phi' \circ m_{\mathbb{A}}$. Then, $m_{\mathcal{B}'} \circ (\phi' \otimes \phi') = \phi' \circ m_{\mathbb{A}}$ and ϕ' is a morphism of \mathcal{R} -algebras. \square

Remark 6.5. Let $R = K$ be a field. Now assume \mathfrak{F} is the family of reflexive functors of \mathcal{K} -modules. Recall Remark 3.13. Proposition 6.4 is likewise true (in this case ϕ' is obviously a morphism of functors of algebras).

Lemma 6.6. Let \mathbb{A} and \mathbb{B} be two reflexive functors of pro-quasicoherent algebras. Let $\{\mathcal{A}_i\}_i$ and $\{\mathcal{B}_j\}_j$ be the quasi-coherent algebra quotients of \mathbb{A} and \mathbb{B} . Then,

$$\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B} = \varprojlim_{i,j} (\mathcal{A}_i \otimes \mathcal{B}_j).$$

Then, $\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$ is a functor of algebras and the natural morphism $\mathbb{A} \otimes \mathbb{B} \rightarrow \mathbb{A} \tilde{\otimes} \mathbb{B}$ is a morphism of functors of algebras.

Proof. We have

$$\begin{aligned} (\mathbb{A}^* \otimes \mathbb{B}^*)^* &= \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}^*, \mathbb{B}) \stackrel{2.13}{=} \mathrm{Hom}_{\mathcal{R}}(\lim_{\substack{\rightarrow \\ i}} \mathcal{A}_i^*, \mathbb{B}) = \mathrm{Hom}_{\mathcal{R}}(\lim_{\substack{\rightarrow \\ i}} \mathcal{A}_i^*, \lim_{\substack{\leftarrow \\ j}} \mathcal{B}_j) \\ &= \lim_{\substack{\leftarrow \\ i,j}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}_i^*, \mathcal{B}_j) \stackrel{2.6}{=} \lim_{\substack{\leftarrow \\ i,j}} (\mathcal{A}_i \otimes \mathcal{B}_j). \end{aligned}$$

□

Theorem 6.7. Let \mathbb{A} and $\mathbb{B} \in \mathfrak{F}$ be two pro-quasicoherent functors of algebras. Then, $\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B} \in \mathfrak{F}$ is a functor of pro-quasicoherent algebras and

$$\mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes_{\mathcal{R}} \mathbb{B}, \mathbb{C}) = \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}, \mathbb{C})$$

for every functor of pro-quasicoherent algebras \mathbb{C} .

Proof. $\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B} = (\mathbb{A}^* \otimes \mathbb{B}^*)^* \in \mathfrak{F}$. Let us follow the notations of Lemma 6.6.

Given a morphism of functor of \mathcal{R} -algebras $\phi: \mathbb{A} \otimes \mathbb{B} \rightarrow \mathbb{C}$, let $\phi_1 = \phi|_{\mathbb{A} \otimes 1}$ and $\phi_2 = \phi|_{1 \otimes \mathbb{B}}$. Then, ϕ_1 factors through an epimorphism onto a quasi-coherent algebra quotient \mathcal{A}_i of \mathbb{A} , and ϕ_2 factors through an epimorphism onto a quasi-coherent algebra quotient \mathcal{B}_j of \mathbb{B} . Then, ϕ factors through $\mathcal{A}_i \otimes \mathcal{B}_j$, and ϕ factors through $\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$. Then,

$$\mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}, \mathbb{C}) \rightarrow \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C})$$

is surjective. It is also injective, because

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}}(\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}, \mathbb{C}) &= \mathrm{Hom}_{\mathcal{R}}(\mathbb{C}^*, (\mathbb{A}^* \otimes \mathbb{B}^*)^{**}) \\ &\stackrel{5.5}{\xrightarrow{}} \mathrm{Hom}_{\mathcal{R}}(\mathbb{C}^*, (\mathbb{A} \otimes \mathbb{B})^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}). \end{aligned}$$

Then, $\mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) = \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}, \mathbb{C})$ for every pro-quasicoherent algebra \mathbb{C} .

A morphism of functors of algebras $f: \mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B} \rightarrow \mathbb{C}$ factors through some $\mathcal{A}_i \otimes \mathcal{B}_j$ because $f|_{\mathbb{A} \otimes \mathbb{B}}$ factors through some $\mathcal{A}_i \otimes \mathcal{B}_j$. Then, the inverse limit of the quasi-coherent algebra quotients of $\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$ is equal to $\lim_{\substack{\leftarrow \\ i,j}} (\mathcal{A}_i \otimes \mathcal{B}_j) = \mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$, that is,

$\mathbb{A} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$ is a pro-quasicoherent algebra.

□

Remark 6.8. Moreover, let $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathfrak{F}$ and \mathbb{C} be pro-quasicoherent functors of \mathcal{R} -algebras, and let $\phi \in \prod_i \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}_i, \mathbb{C}) \subseteq \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}_1 \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \mathbb{A}_n, \mathbb{C})$, then ϕ factors uniquely through $\mathbb{A}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{A}_n$.

Let $\mathbb{A}^* \in \mathfrak{F}$ be a functor of pro-quasicoherent algebras. The multiplication morphism $m: \mathbb{A}^* \otimes \mathbb{A}^* \rightarrow \mathbb{A}^*$ factors uniquely through $\mathbb{A}^* \tilde{\otimes} \mathbb{A}^*$. Then, the comultiplication morphism $\mathbb{A} \rightarrow \mathbb{A} \tilde{\otimes} \mathbb{A}$ factors through $\mathbb{A} \tilde{\otimes} \mathbb{A}$. Taking duals in

$$\mathbb{A}^* \tilde{\otimes} \mathbb{A}^* \tilde{\otimes} \mathbb{A}^* \xrightarrow[\mathrm{Id} \otimes m]{m \otimes \mathrm{Id}} \mathbb{A}^* \tilde{\otimes} \mathbb{A}^* \xrightarrow{m} \mathbb{A}^*,$$

we have that the morphism $\mathbb{A} \rightarrow \mathbb{A} \tilde{\otimes} \mathbb{A}$ is coassociative.

for all functors of procoherent algebras \mathcal{C}^* .

Proof. Let C be a finite dimensional K -algebra and let $\{\mathcal{A}_i\}_{i \in I}$ be the set of all quasi-coherent algebra quotients of \mathbb{A} such that $\dim_K \mathcal{A}_i < \infty$. Then,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}, \mathcal{C}) &\stackrel{6.5}{=} \lim_{\substack{\rightarrow \\ i}} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathcal{A}_i, \mathcal{C}) \stackrel{6.13}{=} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\lim_{\substack{\leftarrow \\ i}} \mathcal{A}_i, \mathcal{C}) \\ &= \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}, \mathcal{C}). \end{aligned}$$

Write $\mathcal{C}^* = \lim_{\substack{\leftarrow \\ j}} \mathcal{C}_j$, where $\{\mathcal{C}_j\}_{j \in J}$ is the set of quasi-coherent algebra quotients of \mathcal{C}^* such that $\dim_K \mathcal{C}_j < \infty$. Then,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}, \mathcal{C}^*) &= \lim_{\substack{\leftarrow \\ j}} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}, \mathcal{C}_j) = \lim_{\substack{\leftarrow \\ j}} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}, \mathcal{C}_j) \\ &= \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}, \mathcal{C}^*). \end{aligned}$$

□

In particular, we have that $\bar{\mathbb{A}} = \mathbb{A}$ and a morphism of functors of \mathcal{K} -algebras $\bar{\mathbb{A}} \rightarrow \mathcal{B}$, where $\dim_K \mathcal{B} < \infty$, is an epimorphism if and only if the obvious morphism $\mathbb{A} \rightarrow \mathcal{B}$ is an epimorphism.

Corollary 6.15. *Let \mathbb{A} be a reflexive functor of \mathcal{K} -algebras. The category of finite \mathcal{K} -dimensional \mathbb{A} -modules is equal to the category of finite \mathcal{K} -dimensional $\bar{\mathbb{A}}$ -modules.*

Proof. Let V be a finite dimensional K -vector space. Then, $\mathrm{End}_K(V)$ is a finite dimensional vector space. By Proposition 6.14

$$\mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}, \mathrm{End}_K(V)) = \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}, \mathrm{End}_K(V)).$$

If \mathcal{V} is an \mathbb{A} -module, then it is naturally an $\bar{\mathbb{A}}$ -module, and reciprocally.

□

Proposition 6.16. *Let $\mathbb{A}_1, \dots, \mathbb{A}_n$ be reflexive functors of \mathcal{K} -algebras. Then,*

$$\overline{\mathbb{A}_1 \otimes_{\mathcal{K}} \dots \otimes_{\mathcal{K}} \mathbb{A}_n} = \bar{\mathbb{A}}_1 \tilde{\otimes}_{\mathcal{K}} \dots \tilde{\otimes}_{\mathcal{K}} \bar{\mathbb{A}}_n.$$

Proof. Let $\{\mathcal{A}_{ij}\}_{j \in J}$ be the set of all quasi-coherent algebra quotients of \mathbb{A}_i such that $\dim_K \mathcal{A}_{ij} < \infty$. Let \mathcal{B} be a finite K -algebra. Any morphism $\mathbb{A}_1 \otimes \dots \otimes \mathbb{A}_n \rightarrow \mathcal{B}$ of functors of \mathcal{K} -algebras factors through some morphism $\mathcal{A}_{1i_1} \otimes \dots \otimes \mathcal{A}_{ni_n} \rightarrow \mathcal{B}$. Likewise, any morphism $\bar{\mathbb{A}}_1 \otimes \dots \otimes \bar{\mathbb{A}}_n \rightarrow \mathcal{B}$ of functors of \mathcal{K} -algebras factors through some morphism $\mathcal{A}_{1i_1} \otimes \dots \otimes \mathcal{A}_{ni_n} \rightarrow \mathcal{B}$. Then,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\overline{\mathbb{A}_1 \otimes_{\mathcal{K}} \dots \otimes_{\mathcal{K}} \mathbb{A}_n}, \mathcal{C}^*) &= \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}_1 \otimes_{\mathcal{K}} \dots \otimes_{\mathcal{K}} \mathbb{A}_n, \mathcal{C}^*) \\ &= \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}_1 \otimes_{\mathcal{K}} \dots \otimes_{\mathcal{K}} \bar{\mathbb{A}}_n, \mathcal{C}^*) = \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}_1 \tilde{\otimes}_{\mathcal{K}} \dots \tilde{\otimes}_{\mathcal{K}} \bar{\mathbb{A}}_n, \mathcal{C}^*) \end{aligned}$$

□

Remark 6.17. *Moreover, let $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathfrak{F}$ and \mathbb{C} be pro-quasicohherent functors of \mathcal{R} -algebras, and let $\phi \in \prod_i \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}_i, \mathbb{C}) \subseteq \mathrm{Hom}_{\mathcal{R}}(\mathbb{A}_1 \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \mathbb{A}_n, \mathbb{C})$, then ϕ factors uniquely through $\mathbb{A}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{A}_n$.*

7. APPLICATIONS TO ALGEBRAIC GEOMETRY

Definition 7.1. Given a functor of commutative \mathcal{R} -algebras \mathbb{A} , the functor $\text{Spec } \mathbb{A}$, “spectrum of \mathbb{A} ”, is defined to be

$$(\text{Spec } \mathbb{A})(S) := \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}, S),$$

for every commutative \mathcal{R} -algebra S .

Proposition 7.2. Let \mathbb{A} be a functor of commutative \mathcal{R} -algebras. Then,

$$\text{Spec } \mathbb{A} = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathcal{R}).$$

Proof. By Adjunction formula (2.9), restricted to the morphisms of algebras, we have that

$$\mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathcal{R})(S) = \text{Hom}_{S\text{-alg}}(\mathbb{A}|_S, S) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}, S) = (\text{Spec } \mathbb{A})(S).$$

□

Therefore, $\text{Spec } \mathbb{A} = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathcal{R}) \subset \mathbb{H}om_{\mathcal{R}}(\mathbb{A}, \mathcal{R}) = \mathbb{A}^*$.

Proposition 7.3. Let \mathbb{X} be a functor of sets and let $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om(\mathbb{X}, \mathcal{R})$ be its functor of functions. Then,

$$\text{Hom}(\mathbb{X}, \text{Spec } \mathbb{B}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{B}, \mathbb{A}_{\mathbb{X}}),$$

for every functor of commutative \mathcal{R} -algebras, \mathbb{B} .

Proof. Given $f: \mathbb{X} \rightarrow \text{Spec } \mathbb{B}$, let $f^*: \mathbb{B} \rightarrow \mathbb{A}_{\mathbb{X}}$ be defined by $f^*(b)(x) := f(x)(b)$, for every $x \in \mathbb{X}$. Given $\phi: \mathbb{B} \rightarrow \mathbb{A}_{\mathbb{X}}$, let $\phi^*: \mathbb{X} \rightarrow \text{Spec } \mathbb{B}$ be defined by $\phi^*(x)(b) := \phi(b)(x)$, for all $b \in \mathbb{B}$. It is easy to check that $f = f^{**}$ and $\phi = \phi^{**}$.

□

Example 7.4. If \mathcal{A} is a commutative \mathcal{R} -algebra, then $\text{Spec } \mathcal{A} = (\text{Spec } \mathcal{A})'$ and $\mathbb{A}_{\text{Spec } \mathcal{A}} = \mathbb{H}om(\text{Spec } \mathcal{A}, \mathcal{R}) = \mathbb{H}om(\text{Spec } \mathcal{A}, \text{Spec } \mathcal{R}[x]) = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathcal{R}[x], \mathcal{A}) = \mathcal{A}$.

Definition 7.5. We will say that a functor of sets \mathbb{X} is affine when $\mathbb{X} = \text{Spec } \mathbb{A}_{\mathbb{X}}$ and $\mathbb{A}_{\mathbb{X}}$ is a reflexive functor of \mathcal{R} -modules.

Let \mathbb{Y} be an affine functor. By Proposition 7.3,

$$(1) \quad \text{Hom}(\mathbb{X}, \mathbb{Y}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}_{\mathbb{Y}}, \mathbb{A}_{\mathbb{X}}).$$

Example 7.6. Affine schemes, $\text{Spec } \mathcal{A}$, are affine functors, by Example 7.4.

Proposition 7.7. We have that $\mathbb{A}_{\lim_i \mathbb{X}_i} = \lim_i \mathbb{A}_{\mathbb{X}_i}$.

Proof. $\mathbb{A}_{\lim_i \mathbb{X}_i} = \mathbb{H}om(\lim_i \mathbb{X}_i, \mathcal{R}) = \lim_i \mathbb{H}om(\mathbb{X}_i, \mathcal{R}) = \lim_i \mathbb{A}_{\mathbb{X}_i}$.

□

Theorem 7.8. Let $\mathbb{A} \in \mathfrak{F}$ be a functor of commutative algebras. Let $\{\mathcal{A}_i\}_i$ be the set of all quasi-coherent algebra quotients of \mathbb{A} . Then,

1. $\text{Spec } \mathbb{A} = \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathcal{R}) \stackrel{6.4}{=} \lim_i \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathcal{A}_i, \mathcal{R}) = \lim_i \text{Spec } \mathcal{A}_i.$
2. $\mathbb{A}_{\text{Spec } \mathbb{A}} \stackrel{7.7}{=} \lim_i \mathcal{A}_i.$

If \mathbb{X} is an affine functor then $\mathbb{A}_{\mathbb{X}}$ is a pro-quasicoherent algebra.

Proof. Assume $\text{Spec } \mathbb{A}$ is affine. Then, $\text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}_{\text{Spec } \mathbb{A}}, \mathcal{B}) = \text{Hom}_{\mathcal{R}\text{-alg}}(\mathbb{A}, \mathcal{B})$. Given a morphism $\mathbb{A}_{\text{Spec } \mathbb{A}} \rightarrow \mathcal{B}$ of functors of \mathcal{R} -algebras, the composite morphism $\mathbb{A} \rightarrow \mathbb{A}_{\text{Spec } \mathbb{A}} \rightarrow \mathcal{B}$ factors through a quotient \mathcal{A}_i , then $\mathbb{A}_{\text{Spec } \mathbb{A}} \rightarrow \mathcal{B}$ factors through \mathcal{A}_i too. Hence, $\{\mathcal{A}_i\}$ is the set of all quasi-coherent algebra quotients of $\mathbb{A}_{\text{Spec } \mathbb{A}}$. Then, $\mathbb{A}_{\text{Spec } \mathbb{A}}$ is a pro-quasicoherent algebra, by 2. \square

Corollary 7.9. *The category of commutative pro-quasicoherent algebras, $\mathbb{A} \in \mathfrak{F}$, is anti-equivalent to the category of affine functors, \mathbb{X} , such that $\mathbb{A}_{\mathbb{X}} \in \mathfrak{F}$. The functors $\mathbb{A} \rightsquigarrow \text{Spec } \mathbb{A}$, $\mathbb{X} \rightsquigarrow \mathbb{A}_{\mathbb{X}}$ establish this anti-equivalence.*

Remark 7.10. *Let $R = K$ be a field. Now, assume \mathfrak{F} is the family of reflexive functors of \mathcal{K} -modules. Recall Remark 6.5. Theorem 7.8 is likewise true. Then, the category of reflexive functors of commutative pro-quasicoherent algebras, is anti-equivalent to the category of affine functors.*

Proposition 7.11. *Let \mathbb{X}, \mathbb{Y} be functors of sets such that $\mathbb{A}_{\mathbb{X}}$ and $\mathbb{A}_{\mathbb{Y}}$ are reflexive functors, then $\mathbb{A}_{\mathbb{X} \times \mathbb{Y}} = \mathbb{A}_{\mathbb{X}} \tilde{\otimes} \mathbb{A}_{\mathbb{Y}}$.*

Proof. $\mathbb{H}om(\mathbb{X} \times \mathbb{Y}, \mathcal{R}) = \mathbb{H}om(\mathbb{X}, \mathbb{H}om(\mathbb{Y}, \mathcal{R})) = \mathbb{H}om(\mathbb{X}, \mathbb{A}_{\mathbb{Y}}) \stackrel{2.18}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{A}_{\mathbb{X}}^*, \mathbb{A}_{\mathbb{Y}}) = (\mathbb{A}_{\mathbb{X}}^* \otimes \mathbb{A}_{\mathbb{Y}}^*)^* = \mathbb{A}_{\mathbb{X}} \tilde{\otimes} \mathbb{A}_{\mathbb{Y}}$. \square

Proposition 7.12. *Let \mathbb{X}, \mathbb{Y} be affine functors such that $\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{Y}} \in \mathfrak{F}$, then $\mathbb{X} \times \mathbb{Y}$ is an affine functor and $\mathbb{A}_{\mathbb{X} \times \mathbb{Y}} \in \mathfrak{F}$.*

Proof. $\mathbb{A}_{\mathbb{X} \times \mathbb{Y}} \stackrel{7.11}{=} \mathbb{A}_{\mathbb{X}} \tilde{\otimes} \mathbb{A}_{\mathbb{Y}} \in \mathfrak{F}$. $\text{Spec } \mathbb{A}_{\mathbb{X} \times \mathbb{Y}} = \text{Spec}(\mathbb{A}_{\mathbb{X}} \tilde{\otimes} \mathbb{A}_{\mathbb{Y}}) \stackrel{6.7}{=} \text{Spec}(\mathbb{A}_{\mathbb{X}} \otimes \mathbb{A}_{\mathbb{Y}}) = \mathbb{X} \times \mathbb{Y}$. \square

7.1. Formal schemes.

Definition 7.13. *Let $\mathcal{C}^* \in \mathfrak{F}$ be a functor of commutative algebras. We will say that $\text{Spec } \mathcal{C}^*$ is a formal scheme. If $\text{Spec } \mathcal{C}^*$ is a functor of monoids we will say that it is a formal monoid.*

Recall that $\mathcal{C}^* \in \mathfrak{F}$ if and only if \mathcal{C} is a free R -module.

Note 7.14. *By Proposition 6.9 and Corollary 7.9, formal schemes are affine functors and $\mathbb{A}_{\text{Spec } \mathcal{C}^*} = \mathcal{C}^*$. Besides, $\text{Spec } \mathcal{C}^*$ is a direct limit of finite R -schemes (see 7.8 and 6.10). Reciprocally, if R is a field, a direct limit of finite R -schemes is a formal scheme, by Theorem 7.16. If R is a field, Demazure ([6]) defines a formal scheme as a functor (from the category of R -finite dimensional rings to sets) which is a direct limit of finite R -schemes.*

The direct product $\text{Spec } \mathcal{C}_1^* \times \text{Spec } \mathcal{C}_2^* = \text{Spec}(\mathcal{C}_1^* \tilde{\otimes} \mathcal{C}_2^*) = \text{Spec}(\mathcal{C}_1 \otimes \mathcal{C}_2)^*$ of formal schemes is a formal scheme.

Theorem 7.15. *Let $\text{Spec } \mathcal{C}^*$ be a formal scheme. Every morphism $\text{Spec } \mathcal{C}^* \rightarrow X = \text{Spec } A$ factors uniquely via $\text{Spec } \mathcal{C}^*$, that is,*

$$\text{Hom}(\text{Spec } \mathcal{C}^*, X) = \text{Hom}_{R\text{-sch}}(\text{Spec } \mathcal{C}^*, X).$$

Proof. By Equation 1,

$$\begin{aligned} \text{Hom}(\text{Spec } \mathcal{C}^*, \text{Spec } A) &= \text{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{A}, \mathcal{C}^*) = \text{Hom}_{R\text{-alg}}(A, \mathcal{C}^*) \\ &= \text{Hom}_{R\text{-sch}}(\text{Spec } \mathcal{C}^*, \text{Spec } A). \end{aligned}$$

\square

Theorem 7.16. *Let $\{\mathrm{Spec} \mathcal{C}_i^*\}_{i \in I}$ be a direct system of formal schemes, or equivalently let $\{\mathcal{C}_i^* \in \mathfrak{F}\}$ be an inverse system of functors of algebras. Write $C = \varinjlim_{i \in I} C_i$,*

then $\mathcal{C}^ = \varprojlim_{i \in I} \mathcal{C}_i^*$. We have that*

$$\varinjlim_i \mathrm{Spec} \mathcal{C}_i^* = \mathrm{Spec}(\varprojlim_i \mathcal{C}_i^*) = \mathrm{Spec} \mathcal{C}^*$$

and $\mathbb{A}_{\mathrm{Spec} \mathcal{C}^} = \mathcal{C}^*$.*

Proof. Observe that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^* \otimes \cdots \otimes \mathcal{C}^*, \mathcal{S}) &= C \otimes \cdots \otimes C \otimes \mathcal{S} = \varinjlim_i (C_i \otimes \cdots \otimes C_i \otimes \mathcal{S}) \\ &= \varinjlim_i \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}_i^* \otimes \cdots \otimes \mathcal{C}_i^*, \mathcal{S}) \end{aligned}$$

Then the kernel of the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^*, \mathcal{S}) \rightarrow \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^* \otimes \mathcal{C}^*, \mathcal{S})$, $f \mapsto \tilde{f}$, where $\tilde{f}(c_1 \otimes c_2) = f(c_1 c_2) - f(c_1)f(c_2)$ coincides with the kernel of the morphism $\varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}_i^*, \mathcal{S}) \rightarrow \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}_i^* \otimes \mathcal{C}_i^*, \mathcal{S})$, $(f_i) \mapsto (\tilde{f}_i)$. Then, $\mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{C}^*, \mathcal{S}) = \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}\text{-alg}}(\mathcal{C}_i^*, \mathcal{S})$ and

$$(\mathrm{Spec} \mathcal{C}^*)(\mathcal{S}) = (\varinjlim_{i \in I} \mathrm{Spec} \mathcal{C}_i^*)(\mathcal{S}).$$

Finally, $\mathbb{A}_{\mathrm{Spec} \mathcal{C}^*} = \varprojlim_i \mathbb{A}_{\mathrm{Spec} \mathcal{C}_i^*} = \varprojlim_i \mathcal{C}_i^* = \mathcal{C}^*$.

□

From now on, in this section, we will assume that $R = K$ is a field.

Definition 7.17. *Let X be a K -scheme and let I be the set of all finite K -subschemes of X . Given K -scheme Y write $A_Y := \mathcal{O}_Y(Y)$, the ring of (regular) functions of Y . Define $\bar{\mathcal{A}}_X := \varprojlim_{i \in I} \mathcal{A}_i$ and*

$$\bar{X} := \mathrm{Spec} \bar{\mathcal{A}}_X \stackrel{7.16}{=} \varinjlim_{i \in I} \mathrm{Spec} \mathcal{A}_i$$

That is, “ \bar{X} is the direct limit of the set of all finite subschemes of X ”.

\bar{X} is a formal scheme and we have a natural monomorphism $\bar{X} \hookrightarrow X^\cdot$.

Theorem 7.18. *Let X be a K -scheme. Then:*

$$\mathrm{Hom}(\mathrm{Spec} \mathcal{C}^*, X^\cdot) = \mathrm{Hom}(\mathrm{Spec} \mathcal{C}^*, \bar{X}),$$

for every formal scheme $\mathrm{Spec} \mathcal{C}^$.*

Proof. $\mathcal{C}^* = \varprojlim_i \mathcal{S}_i$, where the algebras \mathcal{S}_i are finite K -algebras. Then,

$$\begin{aligned} \mathrm{Hom}(\mathrm{Spec} \mathcal{C}^*, X^\cdot) &\stackrel{7.16}{=} \mathrm{Hom}(\varinjlim_i \mathrm{Spec} \mathcal{S}_i, X^\cdot) = \varprojlim_i \mathrm{Hom}(\mathrm{Spec} \mathcal{S}_i, X^\cdot) \\ &= \varprojlim_i \mathrm{Hom}(\mathrm{Spec} \mathcal{S}_i, \bar{X}) = \mathrm{Hom}(\varinjlim_i \mathrm{Spec} \mathcal{S}_i, \bar{X}) = \mathrm{Hom}(\mathrm{Spec} \mathcal{C}^*, \bar{X}). \end{aligned}$$

□

8. FUNCTORS OF BIALGEBRAS

Definition 8.1. A reflexive functor \mathbb{B} of proquasicoherent² algebras is said to be a functor of bialgebras if \mathbb{B}^* is a functor of \mathcal{R} -algebras and the dual morphisms of the multiplication morphism $m: \mathbb{B}^* \otimes \mathbb{B}^* \rightarrow \mathbb{B}^*$ and the unit morphism $u: \mathcal{R} \rightarrow \mathbb{B}^*$ are morphisms of functors of \mathcal{R} -algebras.

Let \mathbb{B}, \mathbb{B}' be two functors of bialgebras. We will say that a morphism of \mathcal{R} -modules, $f: \mathbb{B} \rightarrow \mathbb{B}'$ is a morphism of functors of bialgebras if f and $f^*: \mathbb{B}'^* \rightarrow \mathbb{B}^*$ are morphisms of functors of \mathcal{R} -algebras.

By Proposition 5.4, we can give the following equivalent definition of functor of bialgebras.

Definition 8.2. A reflexive functor \mathbb{B} of proquasicoherent algebras is said to be a functor of bialgebras if it is a functor of coalgebras and the comultiplication morphism $\mathbb{B} \rightarrow \mathbb{B} \tilde{\otimes}_{\mathcal{R}} \mathbb{B}$ and the counit morphism $\mathbb{B} \rightarrow \mathcal{R}$ are morphisms of functors of \mathcal{R} -algebras.

In the literature, an R -algebra B is said to be a bialgebra if it is a coalgebra (with counit) and the comultiplication $c: B \rightarrow B \otimes_R B$ and the counit $e: B \rightarrow R$ are morphisms of R -algebras.

Proposition 8.3. The functors $B \rightsquigarrow \mathcal{B}$ and $\mathcal{B} \rightsquigarrow \mathcal{B}(R)$ establish an equivalence between the category of R -bialgebras and the category of functors of \mathcal{R} -bialgebras.

Proof. Recall 5.1 (1) and 5.4. \square

If \mathcal{B} and $\mathcal{B}^* \in \mathfrak{F}$ are functors of \mathcal{R} -algebras, then they are functors of proquasicoherent algebras, by Proposition 6.9.

Definition 8.4. A functor \mathbb{B} of bialgebras is said to be a functor of pro-quasicoherent bialgebras if \mathbb{B}^* is a functor of pro-quasicoherent algebras.

Theorem 8.5. Let $\mathcal{C}_{\mathfrak{F}-\text{Bialg.}}$ be the category of functors $\mathbb{B} \in \mathfrak{F}$ of pro-quasicoherent bialgebras. The functor $\mathcal{C}_{\mathfrak{F}-\text{Bialg.}} \rightsquigarrow \mathcal{C}_{\mathfrak{F}-\text{Bialg.}}$, $\mathbb{B} \rightsquigarrow \mathbb{B}^*$ is a categorical anti-equivalence.

Proof. Let $\{\mathbb{B}, m, u; \mathbb{B}^*, m', u'\}$ be a functor of pro-quasicoherent bialgebras. Let us only check that $m^*: \mathbb{B}^* \rightarrow (\mathbb{B} \otimes \mathbb{B})^* = \mathbb{B}^* \tilde{\otimes} \mathbb{B}^*$ is a morphism of functors of algebras. By hypothesis, $m'^*: \mathbb{B} \rightarrow (\mathbb{B}^* \otimes \mathbb{B}^*)^* = \mathbb{B} \tilde{\otimes} \mathbb{B}$ is a morphism of functors of algebras. We have the commutative square:

$$(*) \quad \begin{array}{ccc} \mathbb{B} & \xrightarrow{m'^*} & \mathbb{B} \tilde{\otimes} \mathbb{B} \\ m \uparrow & & \uparrow m \otimes m \\ \mathbb{B} \otimes \mathbb{B} & \xrightarrow{m'_{13} \otimes m'_{24}} & \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B} \end{array}$$

where $(m'_{13} \otimes m'_{24})(b_1 \otimes b_2) := \sigma(m'^*(b_1) \otimes m'^*(b_2))$ and $\sigma(b_1 \otimes b_2 \otimes b_3 \otimes b_4) := b_1 \otimes b_3 \otimes b_2 \otimes b_4$, for all $b_i \in \mathbb{B}$. $m': \mathbb{B}^* \otimes \mathbb{B}^* \rightarrow \mathbb{B}^*$ factors through $\mathbb{B}^* \tilde{\otimes} \mathbb{B}^*$, because \mathbb{B}^* is a pro-quasicoherent algebra. Dually, we have the obvious morphisms

$$\mathbb{B} \xrightarrow{m'^*} \mathbb{B} \tilde{\otimes} \mathbb{B} \xrightarrow{I \otimes I} \mathbb{B} \tilde{\otimes} \mathbb{B}$$

and the diagram

²We have assumed \mathbb{B} proquasi-coherent in order that $\mathbb{B} \tilde{\otimes} \mathbb{B}$ be a functor of algebras

$$\begin{array}{ccccc}
\mathbb{B} & \xrightarrow{m'^*} & \mathbb{B} \tilde{\otimes} \mathbb{B} & \xrightarrow{I \otimes I} & \mathbb{B} \tilde{\otimes} \mathbb{B} \\
\uparrow m & & \uparrow m \otimes m & & \uparrow m \otimes m \\
\mathbb{B} \otimes \mathbb{B} & \xrightarrow{m'_{13} \otimes m'_{24}} & \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B} & \xrightarrow{I \otimes I \otimes I \otimes I} & \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B} \tilde{\otimes} \mathbb{B}
\end{array}$$

The right square is commutative and the left square is commutative because the diagram (*) is commutative and $I \otimes I$ is injective (by Proposition 5.5).

Taking duals, in the left square, we obtain the commutative diagram:

$$\begin{array}{ccc}
\mathbb{B}^* & \xleftarrow{m'} & \mathbb{B}^* \tilde{\otimes} \mathbb{B}^* \\
\downarrow m^* & & \downarrow (m \otimes m)^* \\
\mathbb{B}^* \tilde{\otimes} \mathbb{B}^* & \xleftarrow{m'_{13} \otimes m'_{24}} & \mathbb{B}^* \tilde{\otimes} \mathbb{B}^* \tilde{\otimes} \mathbb{B}^* \tilde{\otimes} \mathbb{B}^*
\end{array}$$

which shows that m^* is a morphism of functors of \mathcal{R} -algebras. \square

In [8, Ch. I, §2, 13], Dieudonné proves the anti-equivalence between the category of commutative K -bialgebras and the category of linearly compact cocommutative K -bialgebras (where K is a field).

Remark 8.6. Let $R = K$ be a field and let $\mathcal{C}_{\text{proq-bialg.}}$ be the category of functors of pro-quasicoherent bialgebras. Likewise, the functor $\mathcal{C}_{\text{proq-Bialg.}} \rightsquigarrow \mathcal{C}_{\text{proq-bialg.}}$, $\mathbb{B} \rightsquigarrow \mathbb{B}^*$ is a categorical anti-equivalence.

From now on, in this section, $R = K$ will be a field. Recall Notation 6.12.

Theorem 8.7. Let $\mathbb{B} \in \mathfrak{F}$ be a functor of \mathcal{K} -bialgebras. Then, $\bar{\mathbb{B}} \in \mathfrak{F}$ is a functor of bialgebras and

$$\text{Hom}_{\mathcal{K}\text{-bialg}}(\mathbb{B}, \mathcal{C}^*) = \text{Hom}_{\mathcal{K}\text{-bialg}}(\bar{\mathbb{B}}, \mathcal{C}^*),$$

for all functor of bialgebras \mathcal{C}^* .

Proof. Given any $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathfrak{F}$ pro-quasicoherent algebras then $\overline{\mathbb{A}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{A}_n} = \bar{\mathbb{A}}_1 \tilde{\otimes} \dots \tilde{\otimes} \bar{\mathbb{A}}_n$ because

$$\begin{aligned}
\text{Hom}_{\mathcal{K}\text{-alg}}(\overline{\mathbb{A}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{A}_n}, \mathcal{C}^*) &\stackrel{6.14}{=} \text{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}_1 \tilde{\otimes} \dots \tilde{\otimes} \mathbb{A}_n, \mathcal{C}^*) \\
&\stackrel{6.7}{=} \text{Hom}_{\mathcal{K}\text{-alg}}(\mathbb{A}_1 \otimes \dots \otimes \mathbb{A}_n, \mathcal{C}^*) \stackrel{6.14}{=} \text{Hom}_{\mathcal{K}\text{-alg}}(\overline{\mathbb{A}_1 \otimes \dots \otimes \mathbb{A}_n}, \mathcal{C}^*) \\
&\stackrel{6.16}{=} \text{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathbb{A}}_1 \tilde{\otimes} \dots \tilde{\otimes} \bar{\mathbb{A}}_n, \mathcal{C}^*)
\end{aligned}$$

Then, the comultiplication morphism $\mathbb{B} \rightarrow \mathbb{B} \tilde{\otimes} \mathbb{B}$ defines a comultiplication morphism $\bar{\mathbb{B}} \rightarrow \bar{\mathbb{B}} \tilde{\otimes} \bar{\mathbb{B}}$, and $\bar{\mathbb{B}}$ is a bialgebra scheme.

Given a morphism of functors of bialgebras $f: \mathbb{B} \rightarrow \mathcal{C}^*$, that is, a morphism of functors of algebras such that the diagram

$$\begin{array}{ccc}
\mathbb{B} & \longrightarrow & \mathbb{B} \tilde{\otimes} \mathbb{B} \\
\downarrow f & & \downarrow f \otimes f \\
\mathcal{C}^* & \longrightarrow & \mathcal{C}^* \tilde{\otimes} \mathcal{C}^*
\end{array}$$

is commutative, the induced morphism of functors algebras $\bar{\mathbb{B}} \rightarrow \mathcal{C}^*$ is a morphism of functors of bialgebras. Reciprocally, given a morphism of bialgebras $\bar{\mathbb{B}} \rightarrow \mathcal{C}^*$, the composite morphism $\mathbb{B} \rightarrow \bar{\mathbb{B}} \rightarrow \mathcal{C}^*$ is a morphism of functors of bialgebras. \square

Corollary 8.8. *Let A and B be K -bialgebras. Then,*

$$\mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\bar{\mathcal{A}}, \mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\bar{\mathcal{B}}, \mathcal{A}^*).$$

Proof. It follows from the equalities $\mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\bar{\mathcal{A}}, \mathcal{B}^*) \stackrel{8.7}{=} \mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\mathcal{A}, \mathcal{B}^*) \stackrel{8.5}{=} \mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\mathcal{B}, \mathcal{A}^*) \stackrel{8.7}{=} \mathrm{Hom}_{\mathcal{K}\text{-bialg}}(\bar{\mathcal{B}}, \mathcal{A}^*)$. \square

Note 8.9. *The bialgebra $A^\circ := \mathrm{Hom}_{\mathcal{K}}(\bar{\mathcal{A}}, \mathcal{K})$ is sometimes known as the “dual bialgebra” of A and Corollary 8.8 says (dually) that the functor assigning to each bialgebra its dual bialgebra is autoadjoint (see [1, 3.5]).*

Proposition 8.10. *Let k be a field, L a Lie algebra and $A := U(L)$ the universal enveloping algebra of L . Let $D_A := \{w \in A^* : \dim_k \langle Aw \rangle < \infty\}$ and $G := \mathrm{Spec} D_A$. The category of finite dimensional linear representations of L is equivalent to the category of finite dimensional linear representations of G .*

Proof. The category of finite dimensional linear representations of L is equal to the category of K -finite dimensional A -modules. A is a bialgebra (cocommutative). Let V be a finite dimensional vector space. We have that

$$\mathrm{Hom}_{K\text{-alg}}(A, \mathrm{End}_K(V)) = \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathcal{A}, \mathrm{End}_{\mathcal{K}}(\mathcal{V})) = \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\bar{\mathcal{A}}, \mathrm{End}_{\mathcal{K}}(\mathcal{V})),$$

by Proposition 6.14. Then, the category of finite dimensional linear representations of L is equal to the category of \mathcal{K} -quasicoherent $\bar{\mathcal{A}}$ -modules, \mathcal{V} , such that $\dim_K V < \infty$. This last category is equivalent to the category of finite dimensional linear representations of $\mathrm{Spec} \bar{\mathcal{A}}^*$, by 2.20.

Finally, $\bar{\mathcal{A}}^*$ is the quasi-coherent algebra, \mathcal{D}_A , associated with D_A : $A \rightarrow A_i$ is a K -algebra quotient if and only if A_i is a K -vector space quotient and it is a right and left A -module. Then, $A \rightarrow A_i$ is a K -algebra quotient and $\dim_K A_i < \infty$ if and only if A_i^* is a finite dimensional K -vector subspace of A^* and it is a right and left A -submodule. Hence, $D_A = \lim_{\substack{\rightarrow \\ i}} A_i^*$ and $\mathcal{D}_A = \bar{\mathcal{A}}^*$. \square

8.1. Applications to Algebraic Geometry.

Definition 8.11. *An affine functor $\mathbb{G} = \mathrm{Spec} \mathbb{A}$ is said to be an affine functor of monoids if \mathbb{G} is a functor of monoids.*

Let \mathbb{G} be an affine functor of monoids. Let

$$m: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$$

be the multiplication morphism and let $e \in \mathbb{G}$ the identity element. By Theorem 2.19, $\mathbb{A}_{\mathbb{G}}^*$ is a functor of \mathcal{R} -algebras and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{G} \times \mathbb{G} & \xrightarrow{m} & \mathbb{G} \\ \downarrow & & \downarrow \\ \mathbb{A}_{\mathbb{G}}^* \otimes \mathbb{A}_{\mathbb{G}}^* & \xrightarrow{m} & \mathbb{A}_{\mathbb{G}}^* \end{array}$$

The dual morphisms of the multiplication morphism $m: \mathbb{A}_G^* \otimes \mathbb{A}_G^* \rightarrow \mathbb{A}_G^*$ and the unit morphism $\mathcal{R} \rightarrow \mathbb{A}_G^*$ are the natural morphisms $\mathbb{A}_G \rightarrow \mathbb{A}_{G \times G}$ and $\mathbb{A}_G \xrightarrow{e} \mathcal{R}$, $f \mapsto f(e)$, which are morphisms of \mathcal{R} -algebras. That is, \mathbb{A}_X is a functor of bialgebras.

Let X be an affine functor and assume \mathbb{A}_X is a functor of bialgebras. Let $m: \mathbb{A}_X^* \otimes \mathbb{A}_X^* \rightarrow \mathbb{A}_X^*$ and $e: \mathcal{R} \rightarrow \mathbb{A}_X^*$ the multiplication and unit morphisms. Given a point $(x, x') \in X \times X \subset \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}_{X \times X}, \mathcal{R})$ then $(x, x') \circ m^* \in \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}_X, \mathcal{R}) = X$ and we have the commutative diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{m} & X \\ \downarrow & & \downarrow \\ \mathbb{A}_X^* \otimes \mathbb{A}_X^* & \xrightarrow{m} & \mathbb{A}_X^* \end{array}$$

Obviously $e \in \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}_X, \mathcal{R}) = X$. It is easy to check that $\{X, m, e\}$ is a functor of monoids.

Proposition 8.12. *Let G and G' be affine functors of monoids. Then,*

$$\mathbb{H}om_{\text{mon}}(G, G') = \mathbb{H}om_{\mathcal{R}\text{-bialg}}(\mathbb{A}_{G'}, \mathbb{A}_G).$$

Proof. Let $h: G \rightarrow G'$ be a morphism of functors of monoids. The composition morphism of h with the natural morphism $G' \rightarrow \mathbb{A}_{G'}$ factors through \mathbb{A}_G^* , that is, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{h} & G' \\ \downarrow & & \downarrow \\ \mathbb{A}_G^* & \longrightarrow & \mathbb{A}_{G'}^* \end{array}$$

The dual morphism $\mathbb{A}_{G'} \rightarrow \mathbb{A}_G$ is the morphism induced by h between the functors of functions. Conversely, let $f: \mathbb{A}_{G'} \rightarrow \mathbb{A}_G$ be a morphism of functors of \mathcal{R} -algebras, such that f^* is also a morphism of functors of \mathcal{R} -algebras. Given $g \in G$, then $f^*(g) = g \circ f \in \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}_{G'}, \mathcal{R}) = G'$. Hence, $f|_G: G \rightarrow G'$ is a morphism of functors of monoids. \square

Theorem 8.13. *The category of affine functors of monoids is anti-equivalent to the category of functors of commutative bialgebras.*

Theorem 8.14. *The category of cocommutative bialgebras A is equivalent to the category of formal monoids $\text{Spec } \mathcal{A}^*$ (we assume the R -modules A are free).*

In [8, Ch. I, §2, 14], it is given the Cartier Duality (formal schemes are certain functors over the category of commutative linearly compact algebras over a field).

Definition 8.15. *Let G be a functor of abelian monoids. $G^\vee := \mathbb{H}om_{\text{mon}}(G, \mathcal{R})$ (where we regard \mathcal{R} as a monoid with the operation of multiplication) is said to be the dual monoid of G .*

If G is a functor of groups, then $G^\vee = \mathbb{H}om_{\text{grp}}(G, G_m)$ ($G_m := \text{Spec } R[x, 1/x]$).

Theorem 8.16. *Let G be a functor of abelian monoids with a reflexive functor of functions. Then, $G^\vee = \text{Spec } (\mathbb{A}_G^*)$ (in particular, this equality shows that $\text{Spec } \mathbb{A}_G^*$ is a functor of abelian monoids).*

Proof. $G^\vee = \mathbb{H}om_{\text{mon}}(G, \mathcal{R}) \stackrel{2.19}{=} \mathbb{H}om_{\mathcal{R}\text{-alg}}(\mathbb{A}_G^*, \mathcal{R}) = \text{Spec } (\mathbb{A}_G^*)$. \square

Theorem 8.17. *The category of abelian affine R -monoid schemes $G = \operatorname{Spec} A$ is anti-equivalent to the category of abelian formal monoids $\operatorname{Spec} \mathcal{A}^*$ (we assume the R -modules A are free). The functor $\mathbb{G} \rightsquigarrow \mathbb{G}^\vee$ gives the categorical anti-equivalence.*

Let X and Y be two K -schemes. By the universal property 7.18, it can be checked that

$$\overline{X \times Y} = \bar{X} \times \bar{Y}.$$

Theorem 8.18. *Let G be a K -scheme on groups (resp. monoids). Then \bar{G} is a functor of groups (resp. monoids), the natural morphism $\bar{G} \rightarrow G$ is a morphism of functors of monoids and*

$$\operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} \mathcal{C}^*, G) = \operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} \mathcal{C}^*, \bar{G}),$$

for every formal monoid $\operatorname{Spec} \mathcal{C}^*$. If G is commutative, then \bar{G} is commutative.

Proof. Let $\mu: G \times G \rightarrow G$ the multiplication morphism. By Theorem 7.18, the composite morphism $\overline{G \times G} = \bar{G} \times \bar{G} \rightarrow G \times G \rightarrow G$ factors through a unique morphism $\mu': \bar{G} \times \bar{G} \rightarrow \bar{G}$, that is, we have the commutative diagram:

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \longrightarrow & G \times G \\ \downarrow \mu' & & \downarrow \mu \\ \bar{G} & \longrightarrow & G \end{array}$$

Let $*$: $G \rightarrow G$ be the inverse morphism. The composition $\bar{G} \rightarrow G \xrightarrow{*} G$ factors through a unique morphism $*': \bar{G} \rightarrow \bar{G}$, that is, we have the commutative diagram:

$$\begin{array}{ccc} \bar{G} & \longrightarrow & G \\ \downarrow *' & & \downarrow * \\ \bar{G} & \longrightarrow & G \end{array}$$

Now it is easy to check that $(\bar{G}, \mu', *')$ is a functor of groups and to conclude the proof. \square

Proposition 8.19. *Let $\operatorname{Spec} \mathcal{C}^*$ be a formal monoid and $D_{\mathcal{C}} = \{w \in \mathcal{C}^* : w(I) = 0 \text{ for some bilateral ideal } I \subset \mathcal{C} \text{ of finite codimension}\} \subset \mathcal{C}^*$. Then, $\operatorname{Spec} D_{\mathcal{C}}$ is an affine monoid scheme and*

$$\operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} \mathcal{C}^*, \operatorname{Spec} A) = \operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} D_{\mathcal{C}}, \operatorname{Spec} A),$$

for every affine monoid scheme $\operatorname{Spec} A$.

Proof. Observe that $D_{\mathcal{C}} = \varinjlim_I (C/I)^*$ and $\mathcal{D}_{\mathcal{C}}^* = \bar{\mathcal{C}}$. Then,

$$\begin{aligned} \operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} \mathcal{C}^*, \operatorname{Spec} A) &\stackrel{8.13}{=} \operatorname{Hom}_{\operatorname{bialg}}(\mathcal{A}, \mathcal{C}^*) \stackrel{8.5}{=} \operatorname{Hom}_{\operatorname{bialg}}(\mathcal{C}, \mathcal{A}^*) \\ &\stackrel{6.14}{=} \operatorname{Hom}_{\operatorname{bialg}}(\bar{\mathcal{C}}, \mathcal{A}^*) \stackrel{8.5}{=} \operatorname{Hom}_{\operatorname{bialg}}(\mathcal{A}, \mathcal{D}_{\mathcal{C}}) \stackrel{8.13}{=} \operatorname{Hom}_{\operatorname{mon}}(\operatorname{Spec} D_{\mathcal{C}}, \operatorname{Spec} A). \end{aligned}$$

\square

Note 8.20. *Let X be a K -scheme and A_X the ring of functions of X . The set D_X of distributions of X of finite support is said to be $D_X := \{w \in A_X^* : w \text{ factors through a finite quotient algebra of } A_X\}$. Obviously, $\mathcal{D}_X^* = \bar{A}_X$ and $\operatorname{Spec} \mathcal{D}_X^* = \bar{X}$.*

If $\text{Spec } \mathcal{C}^*$ is an abelian formal monoid then $G = \text{Spec } \mathcal{C}$ is an affine abelian monoid scheme and $D_{\mathcal{C}} = D_G$, then

$$(2) \quad \text{Hom}_{\text{mon}}(G^{\vee}, \text{Spec } A) = \text{Hom}_{\text{mon}}(\text{Spec } D_G, \text{Spec } A),$$

for every affine monoid scheme $\text{Spec } A$.

Assume $G = \text{Spec } A$ and $G' = \text{Spec } B$ are commutative affine monoid schemes, then

$$(3) \quad \begin{aligned} \text{Hom}_{\text{mon}}(\text{Spec } D_G, G') &\stackrel{\text{Eq. 2}}{=} \text{Hom}_{\text{mon}}(G^{\vee}, G') = \text{Hom}_{\text{mon}}(G'^{\vee}, G) \\ &\stackrel{\text{Eq. 2}}{=} \text{Hom}_{\text{mon}}(\text{Spec } D_{G'}, G). \end{aligned}$$

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(Pedro Sancho) DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS S/N, 06071 BADAJOZ, SPAIN
E-mail address: `sancho@unex.es`